

EXISTENCE OF NONTRIVIAL LOGARITHMIC CO-HIGGS STRUCTURE ON CURVES

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ABSTRACT. We study various aspects on nontrivial logarithmic co-Higgs structure associated to unstable bundles on algebraic curves. We check several criteria for (non-)existence of nontrivial logarithmic co-Higgs structures and describe their parameter spaces. We also investigate the Segre invariants of these structures and see their non-simplicity. In the end we also study the higher dimensional case, specially when the tangent bundle is not semistable.

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1. INTRODUCTION

A logarithmic co-Higgs sheaf on a complex manifold X is a pair (\mathcal{E}, Φ) with a torsion-free coherent sheaf \mathcal{E} on X and a morphism $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ satisfying the integrability condition $\Phi \wedge \Phi = 0$, where $\mathcal{T}_{\mathcal{D}}$ is the logarithmic tangent bundle X associated to an arrangement \mathcal{D} of hypersurfaces with simple normal crossings.

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When \mathcal{D} is empty, it is a co-Higgs sheaf in the usual sense, introduced and developed by Hitchin and Gualtieri; see [18, 15]. When \mathcal{E} is locally free, it is a generalized vector bundle on X considered as a generalized complex manifold, whose co-Higgs field vanishes in the normal direction to the support of \mathcal{D} .

It is observed in [4, Theorem 1.1] that the semistability of a co-Higgs bundle (\mathcal{E}, Φ) on X with nonnegative Kodaira dimension implies the semistability of \mathcal{E} . In case of negative Kodaira dimension, there are several works on description of moduli space of semistable co-Higgs bundles, including the case when the associated bundle is not stable; see [27] and [10].

Now the additional condition for a co-Higgs field to vanish in the normal direction to \mathcal{D} with higher degree, forces the associated bundle to be unstable. So we are mainly interested in the logarithmic co-Higgs sheaves associated to the arrangement with high degree and assume that the length of Harder-Narasimhan filtration is at least two. We fix numeric data for the Harder-Narasimhan filtration of the sheaf in consideration, i.e. fix the length s at least two of the filtration together with rank r_i and degree d_i of the successive quotients in the Harder-Narasimhan filtration (2). Setting $\gamma := \deg \mathcal{T}_{\mathcal{D}}$ and $\mu_i := d_i/r_i$, we always assume that $\mu_s - \mu_1 \leq \gamma < 0$ as the least requirement for the existence of the non-trivial co-Higgs field; see Corollary 3.9. Then we investigate the numeric criterion for the sheaf to admit a non-trivial co-Higgs field; see Proposition 3.4 and Theorem 3.7.

Theorem 1.1. *Fix the numeric data for the Harder-Narasimhan filtration and denote by \mathbb{U} the set of the torsion-free sheaves on an algebraic curve X with these data. Then the following hold:*

- (i) *there exists an unstable sheaf in \mathbb{U} with non-trivial co-Higgs field;*
- (ii) *the inequality $\mu_s - \mu_1 \geq \gamma + 1 - g$ implies the existence of an unstable sheaf in \mathbb{U} with no non-trivial co-Higgs field;*
- (iii) *the inequality $\mu_s - \mu_1 < \gamma + 1 - g$ implies that every sheaf in \mathbb{U} admits a non-trivial co-Higgs field.*

The existence part is induced by explicit usage of positive elementary transformations and the positive answer to the Lange conjecture [28]. Furthermore we extend the notion of Segre invariant to the setting of logarithmic co-Higgs sheaves and show that it is well-defined over curves under the assumption that $\gamma < 0$ and that this invariant is same as the usual Segre invariant under a certain condition; see Corollary 4.8 and Proposition 4.14.

Theorem 1.2. *For a logarithmic co-Higgs sheaf (\mathcal{E}, Φ) on an algebraic curve X with $\gamma < 0$, the k^{th} -Segre invariant $s_k(\mathcal{E}, \Phi)$ is well-defined. It is also equal to the Segre invariant $s_k(\mathcal{E})$ in the usual sense, if \mathcal{E} admits the complete Harder-Narasimhan filtration, i.e. $r_i = 1$ for all i .*

Then we check in Proposition 4.14 that co-Higgs sheaves associated to unstable bundle are usually not stable, not even simple.

Over algebraic curves the bundle $\mathcal{T}_{\mathcal{D}}$ is automatically semistable. So, as the counterpart to the case of algebraic curves, in §5 we deal with the case when the dimension of X is at least 2 and $\mathcal{T}_{\mathcal{D}}$ is not semistable. Under the assumption that the biggest slope in the Harder-Narasimhan filtration of $\mathcal{T}_{\mathcal{D}}$ is negative, we give a recipe to construct all the pairs (\mathcal{E}, Φ) with \mathcal{E} reflexive of $\text{rk}(\mathcal{E}) = r \in \{2, 3\}$ and non-trivial co-Higgs field $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. When $r = 2$ and in most cases with $r = 3$,

the map Φ is always 2-nilpotent and so it is integrable. We also point out exactly when we cannot guarantee the integrability.

2. PRELIMINARY

Let X be a smooth projective variety of dimension n at least one with the tangent bundle T_X over the field of complex numbers \mathbb{C} . We fix an ample line bundle $\mathcal{O}_X(1)$ and denote by $\mathcal{E}(t)$ the twist of \mathcal{E} by $\mathcal{O}_X(t)$ for any coherent sheaf \mathcal{E} on X and $t \in \mathbb{Z}$. We also denote by \mathcal{E}^\vee the dual of \mathcal{E} . The dimension of cohomology group $H^i(X, \mathcal{E})$ is denoted by $h^i(X, \mathcal{E})$ and we will skip X in the notation, if there is no confusion. We define the slope $\mu(\mathcal{E})$ of a coherent sheaf \mathcal{E} on X with respect to $\mathcal{O}_X(1)$ to be $\deg \mathcal{E} / \text{rk}(\mathcal{E})$.

Now consider an arrangement $\mathcal{D} = \{D_1, \dots, D_m\}$ of pairwise distinct, smooth and irreducible divisors D_i on X , and if there is no confusion we also denote by \mathcal{D} the divisor $D_1 + \dots + D_m$. We assume that the divisor \mathcal{D} has simple normal crossings. Then the associated logarithmic tangent bundle $T_X(-\log \mathcal{D})$ is locally free and fits into the following exact sequence; see [13].

$$(1) \quad 0 \rightarrow T_X(-\log \mathcal{D}) \rightarrow T_X \rightarrow \bigoplus_{i=1}^m \varepsilon_{i*} \mathcal{O}_{D_i}(D_i) \rightarrow 0,$$

where $\varepsilon_i : D_i \rightarrow X$ is the embedding. If there is no confusion, we will simply denote $T_X(-\log \mathcal{D})$ by $\mathcal{T}_{\mathcal{D}}$.

Definition 2.1. [4] A \mathcal{D} -logarithmic co-Higgs sheaf on X is a pair (\mathcal{E}, Φ) where \mathcal{E} is a torsion-free coherent sheaf on X and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with $\Phi \wedge \Phi = 0$. Here Φ is called the *logarithmic co-Higgs field* of (\mathcal{E}, Φ) and the condition $\Phi \wedge \Phi = 0$ is an integrability condition originating in the work of Simpson [29].

For a torsion-free coherent sheaf \mathcal{E} on X , we consider its associated Harder-Narasimhan filtration:

$$(2) \quad \{0\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_s = \mathcal{E}$$

with the graduation $gr(\mathcal{E}) := \bigoplus_{i=1}^s \mathcal{F}_i / \mathcal{F}_{i-1}$ such that each $\mathcal{F}_i / \mathcal{F}_{i-1}$ is semistable and $\mu(\mathcal{F}_i / \mathcal{F}_{i-1})$ is strictly decreasing for all $i < s$. The integer s is called the *length* of the filtration, and if $s = r$, then the filtration is said to be *complete*. We denote by $\mu_+(\mathcal{E})$ and $\mu_-(\mathcal{E})$ the maximal and minimal slopes in the filtration, respectively:

$$\mu_+(\mathcal{E}) := \mu(\mathcal{F}_1), \quad \mu_-(\mathcal{E}) := \mu(\mathcal{F}_s / \mathcal{F}_{s-1}).$$

Remark 2.2. For two torsion-free sheaves \mathcal{A} and \mathcal{B} on X , let $\mathcal{A} \overline{\otimes} \mathcal{B}$ be the quotient of $\mathcal{A} \otimes \mathcal{B}$ by its torsion. If \mathcal{A} and \mathcal{B} are semistable, then $\mathcal{A} \overline{\otimes} \mathcal{B}$ is also semistable by [22, Theorem 2.5]. Applying this observation to the Harder-Narasimhan filtrations of \mathcal{A} and \mathcal{B} , we get that $\mu_+(\mathcal{A} \overline{\otimes} \mathcal{B}) = \mu_+(\mathcal{A}) + \mu_+(\mathcal{B})$.

Lemma 2.3. *If $f : \mathcal{A} \rightarrow \mathcal{B}$ is a nonzero map between two torsion-free sheaves on X , then we have $\mu_-(\mathcal{A}) \leq \mu_+(\mathcal{B})$*

Proof. Let $\{0\} = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_a = \mathcal{A}$ be the Harder-Narasimhan filtration of \mathcal{A} and let $k \in \{1, \dots, a\}$ be the minimal integer such that $\mathcal{A}_k \not\subseteq \ker(f)$, i.e. the minimal integer such that $f|_{\mathcal{A}_k} \neq 0$. Then we have $f|_{\mathcal{A}_{k-1}} \equiv 0$ and so $f|_{\mathcal{A}_k}$ induces a nonzero map $\tilde{f} : \mathcal{A}_k / \mathcal{A}_{k-1} \rightarrow \mathcal{B}$.

Let $\{0\} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \dots \subset \mathcal{B}_b = \mathcal{B}$ be the Harder-Narasimhan filtration of \mathcal{B} and let l be the minimal positive integer $l \leq b$ such that $\tilde{f}(\mathcal{A}_k / \mathcal{A}_{k-1}) \subseteq \mathcal{B}_l$. Then we have $\tilde{f}(\mathcal{A}_k / \mathcal{A}_{k-1}) \not\subseteq \mathcal{B}_{l-1}$ and so \tilde{f} induces a nonzero map $\hat{f} : \mathcal{A}_k / \mathcal{A}_{k-1} \rightarrow \mathcal{B}_l / \mathcal{B}_{l-1}$.

Since $\mathcal{A}_k/\mathcal{A}_{k-1}$ and $\mathcal{B}_l/\mathcal{B}_{l-1}$ are semistable, we have $\mu(\mathcal{A}_k/\mathcal{A}_{k-1}) \leq \mu(\mathcal{B}_l/\mathcal{B}_{l-1})$. By the definition of μ_+ and μ_- in terms of the Harder-Narasimhan filtration we have $\mu(\mathcal{A}_k/\mathcal{A}_{k-1}) \geq \mu_-(\mathcal{A})$ and $\mu(\mathcal{B}_l/\mathcal{B}_{l-1}) \leq \mu_+(\mathcal{B})$, concluding the assertion. \square

Remark 2.2 and Lemma 2.3 give the following whose assertion will be assumed throughout this article.

Corollary 2.4. *Assuming the existence of a nonzero map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, we have*

$$(3) \quad \mu_-(\mathcal{E}) \leq \mu_+(\mathcal{E}) + \mu_+(\mathcal{T}_{\mathcal{D}}) = \mu_+(\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}).$$

Remark 2.5. Assume that \mathcal{E} is not semistable and so $s \geq 2$. If there exists a nonzero map $f \in \text{Hom}(\mathcal{E}/\mathcal{F}_{s-1}, \mathcal{F}_{s-1} \otimes \mathcal{T}_{\mathcal{D}})$, then we may composite the quotient map $\mathcal{E} \rightarrow \mathcal{E}/\mathcal{F}_{s-1}$ with it to get a nonzero 2-nilpotent logarithmic co-Higgs field Φ_f . Note that the associated co-Higgs field is uniquely determined by the choice of a map, i.e. if f and g are two different nonzero maps in $\text{Hom}(\mathcal{E}/\mathcal{F}_{s-1}, \mathcal{F}_{s-1} \otimes \mathcal{T}_{\mathcal{D}})$, then we get $\Phi_f \neq \Phi_g$.

If n is at least two, we fix a polarization $\mathcal{O}_X(1)$ with respect to which we consider (semi-)stability. For most cases in this article we will mainly assume that \mathcal{D} is of high degree so that $\mathcal{T}_{\mathcal{D}}$ is “sufficiently negative” and that $\mathcal{T}_{\mathcal{D}}$ is semistable with $\gamma = \deg \mathcal{T}_{\mathcal{D}} < 0$, except in §5.2;

Assumption 2.6. *We always assume that $\gamma = \deg \mathcal{T}_{\mathcal{D}}$ is negative, if there is no specification.*

Remark 2.7. There are manifolds with Ω_X^1 ample as in [11, 12], in which cases we may even take $\mathcal{D} = \emptyset$: If instead of logarithmic co-Higgs field we use the field $T_X(-\mathcal{D}) \cong T_X \otimes \mathcal{O}_X(-\mathcal{D})$ vanishing on a divisor \mathcal{D} , then we may use the semistability of the tangent bundle of many Fano manifolds [26] and then take a very positive \mathcal{D} to get $T_X(-\mathcal{D})$ negative and semistable.

We fix a triple of integers $(r, d, s) \in \mathbb{Z}^3$ together with pairs $(r_i, d_i) \in \mathbb{Z}^2$ for $1 \leq i \leq s$ such that $r \geq 2$, $s \geq 1$, $r_i \geq 1$ and

$$r = r_1 + \cdots + r_s, \quad d = d_1 + \cdots + d_s.$$

Assume further that $d_i/r_i > d_{i+1}/r_{i+1}$ for $i = 1, \dots, s-1$. Then we denote by

$$\mathbb{U} = \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$$

the set of all torsion-free coherent sheaves \mathcal{E} of rank r such that the Harder-Narasimhan filtration (2) of \mathcal{E} with respect to $\mathcal{O}_X(1)$ has $(r_1, d_1; \dots; r_s, d_s)$ as its numerical data, i.e. each quotient sheaf $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable of rank r_i and degree d_i . By [22] the filtration (2) tensored by $\mathcal{T}_{\mathcal{D}}$

$$(4) \quad \{0\} = \mathcal{F}_0 \otimes \mathcal{T}_{\mathcal{D}} \subset \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}} \subset \cdots \subset \mathcal{F}_s \otimes \mathcal{T}_{\mathcal{D}}$$

is the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ if $\mathcal{T}_{\mathcal{D}}$ is semistable. We also assume the existence of a nonzero co-Higgs field $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$: if n is at least two, we do not assume for the moment the integrability condition $\Phi \wedge \Phi = 0$, because in the most examples in this article it will follow from the other assumption, or from Lemma 2.8, where we assume that s is at least two.

Denote by $\tilde{\Phi}$ the following map:

$$\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}^{\otimes 2}$$

induced by Φ . Comsiting the natural map $\mathcal{T}_D^{\otimes 2} \rightarrow \wedge^2 \mathcal{T}_D$ with $\widetilde{\Phi} \circ \Phi$, we have $\Phi \wedge \Phi$ as an element in $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \wedge^2 \mathcal{T}_D)$.

Lemma 2.8. *If s is at least two, then we have $\widetilde{\Phi} \circ \Phi = 0$, i.e. Φ is 2-nilpotent. In particular, we have $\Phi \wedge \Phi = 0$.*

Proof. Since we assume that γ is negative, the sheaf $\mathcal{F}_1 \otimes \mathcal{T}_D \subset \mathcal{E} \otimes \mathcal{T}_D$ is the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_D$ and $\mathcal{F}_1 \otimes \mathcal{T}_D^{\otimes 2} \subset \mathcal{E} \otimes \mathcal{T}_D^{\otimes 2}$ is the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_D^{\otimes 2}$. Thus we have $\Phi(\mathcal{E}) \subseteq \mathcal{F}_1 \otimes \mathcal{T}_D$ and $\widetilde{\Phi}(\mathcal{F}_1 \otimes \mathcal{T}_D) = 0$, implying that $\widetilde{\Phi} \circ \Phi = 0$. \square

3. CURVE CASE

Assume that X is a smooth algebraic curve of genus g and take $\mathcal{D} = \{p_1, \dots, p_m\}$ a set of m distinct points. Then we have $\mathcal{T}_D \cong T_X \otimes \mathcal{O}_X(-\mathcal{D})$ with degree $\gamma := 2 - 2g - m$. We assume that γ is negative so that we are not in the set-up of [24]. The sequence (1) turns into the following

$$0 \rightarrow \mathcal{T}_D \rightarrow T_X \rightarrow \bigoplus_{i=1}^m \mathbb{C}_{p_i} \rightarrow 0.$$

Another feature of the case $n = 1$ is that all logarithmic co-Higgs fields automatically satisfy the integrability condition.

Consider a vector bundle \mathcal{E} of rank r with the Harder-Narasimhan filtration (2) and we assume

$$(5) \quad \mu_-(\mathcal{E}) = \mu(\mathcal{F}_s/\mathcal{F}_{s-1}) \leq \gamma + \mu(\mathcal{F}_1) = \gamma + \mu_+(\mathcal{E}),$$

which is a necessary condition for the existence of a nonzero map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_D$; see Corollary 3.9.

Remark 3.1. For each $i \in \{1, \dots, s\}$ with $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \gamma \geq \mu_-(\mathcal{E})$, define

$$b(i) := \min_{i+1 \leq k \leq s} \{k \mid \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \gamma \geq \mu(\mathcal{F}_k/\mathcal{F}_{k-1})\}$$

and then the map Φ induces a map $\Phi^i : \mathcal{F}_{b(i)}/\mathcal{F}_{b(i)-1} \rightarrow (\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes \mathcal{T}_D$. Similarly, for each $j \in \{2, \dots, s\}$ with $\mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \leq \gamma + \mu_+(\mathcal{E})$, define

$$c(j) := \max_{1 \leq k \leq j-1} \{k \mid \mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \leq \gamma + \mu(\mathcal{F}_k/\mathcal{F}_{k-1})\}.$$

The map Φ induces a map $\Phi_j : \mathcal{F}_j/\mathcal{F}_{j-1} \rightarrow (\mathcal{F}_{c(j)}/\mathcal{F}_{c(j)-1}) \otimes \mathcal{T}_D$. Note that these maps Φ^i and Φ_j are not necessarily nonzero.

Now fix the following numeric data

$$(s; r_1, \dots, r_s; d_1, \dots, d_s) \in \mathbb{Z}^{\oplus (2s+1)}$$

with $s, r_i > 0$ for each i such that $d_i/r_i > d_{i+1}/r_{i+1}$ for all i ; if $g = 0$, we also assume $a_i/r_i \in \mathbb{Z}$ for each i . Recall that we denote by $\mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ the set of all vector bundles \mathcal{E} of rank $r := \sum_{i=1}^s r_i$ on X with the Harder-Narasimhan filtration (2) such that $\text{rk}(\mathcal{F}_i/\mathcal{F}_{i-1}) = r_i$ and $\deg \mathcal{F}_i/\mathcal{F}_{i-1} = d_i$ for each i . The conditions just given above for s, r_i and d_i are the necessary and sufficient conditions for the existence of a vector bundle \mathcal{E} on X with rank r and degree $d := d_1 + \dots + d_s$.

Indeed, for the existence part, in case $g \geq 2$ we may even take a stable bundle $\mathcal{F}_i/\mathcal{F}_{i-1}$, while in case $g = 1$ by Atiyah's classification of vector bundles on elliptic

curves, we may take as $\mathcal{F}_i/\mathcal{F}_{i-1}$ a semistable bundle; we can choose either indecomposable one or polystable one, depending on our purpose.

To get parameters spaces we first get parameter spaces for the sheaves \mathcal{E} , then for a fixed sheaf \mathcal{E} we study all logarithmic co-Higgs fields $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ and then we put together the informations. We have several problems coming from the sheaves \mathcal{E} such as non-separatedness or often reducibility of moduli of sheaves, and then more problems bring the logarithmic co-Higgs field into the picture.

First of all, we fix enough numerical invariant to get a bounded family of pairs (\mathcal{E}, Φ) . Fixing an ample line bundle $\mathcal{O}_X(1)$, we consider sheaves \mathcal{E} with a Harder-Narasimhan filtration (2) and we fix the Hilbert function of each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$. Since each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is assumed to be semistable, the family of all $\mathcal{F}_i/\mathcal{F}_{i-1}$ are bounded. We first see that the Ext^1 -groups involved in the extensions

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_2/\mathcal{F}_1 \rightarrow 0$$

are upper bounded and that the set of all \mathcal{F}_2 is bounded. Then we consider the set of all \mathcal{F}_3 and so on, inductively. We may get relative Ext^1 -groups as parameter spaces, but these parameter spaces usually do not parametrizes one-to-one isomorphism classes of sheaves, even by taking into account that proportional extensions gives isomorphic sheaves.

For the relative Ext^1 we need to have universal family parametrizing all $\mathcal{F}_i/\mathcal{F}_{i-1}$ and we usually need to work with parameter spaces of sheaves which do not parametrizes one-to-one isomorphism classes. Note that there is a flat family with isomorphic sheaves \mathcal{E} whose flat limit is $\text{gr}(\mathcal{E}) = \bigoplus_{i=1}^s \mathcal{F}_i/\mathcal{F}_{i-1}$. Thus there is no hope of one-to-one parametrization of isomorphism classes of sheaves; when the numerology allows that some $\mathcal{F}_i/\mathcal{F}_{i-1}$ is strictly semistable, then this phenomenon occurs even for the graded subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$. Algebraic stacks of course do not parametrize isomorphism classes of sheaves, not even of vector bundles; see [14]. In the case $n = 1$ with $X = \mathbb{P}^1$, we have a unique bundle, \mathcal{E} for any fixed parameter space $\mathbb{U}_{\mathbb{P}^1}(s; r_1, d_1; \dots; r_s, d_s)$ and so the parameter space for (\mathcal{E}, Φ) is the vector space $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}})$, which parametrizes one-to-one the isomorphism classes of pairs (\mathcal{E}, Φ) . See Remark 3.12 for the case $n = 1$ and X a curve of genus $g \geq 2$.

Remark 3.2. In the case $s = 2$, the datum of (\mathcal{E}, Φ) with $[\mathcal{E}] \in \mathbb{U}_X(2; r_1, d_1; r_2, d_2)$ and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ induces a holomorphic triple $\psi : \mathcal{E}/\mathcal{F}_1 \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ in the sense of [9] and we may study the stability of the holomorphic triple. Conversely, for every holomorphic triple $f : \mathcal{G}_2 \rightarrow \mathcal{G}_1 \otimes \mathcal{T}_{\mathcal{D}}$ such that \mathcal{G}_1 and \mathcal{G}_2 are semistable with $\text{rk}(\mathcal{G}_i) = r_i$ and $\deg \mathcal{G}_i = d_i$, $i = 1, 2$, and for any extension class

$$(6) \quad 0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{G}_2 \rightarrow 0,$$

we get $[\mathcal{E}] \in \mathbb{U}_X(2; r_1, d_1; r_2, d_2)$ with $0 \subset \mathcal{G}_1 \subset \mathcal{E}$ as its Harder-Narasimhan filtration and a 2-nilpotent map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ induced by f . Two sheaves, say \mathcal{E} and \mathcal{E}' , fitting as middle bundles in (6) for the same \mathcal{G}_1 and \mathcal{G}_2 are isomorphic if and only if their associated extensions are proportional, because \mathcal{G}_1 and \mathcal{G}_2 are assumed to be semistable with $d_1/r_1 > d_2/r_2$ and so (6) is the Harder-Narasimhan filtration of the bundle in the middle.

This argument fits very well in §5.1, where $\mathcal{T}_{\mathcal{D}}$ is assumed to be semistable, because $\mathcal{F}_i \otimes \mathcal{T}_{\mathcal{D}}$ would be in the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$; in this case we only require that \mathcal{E} is torsion-free and then define $\mathbb{U}(s; r_1, d_1; \dots; r_s, d_s)$ with Mumford's (slope-)semistability.

3.1. Projective line. We take $X = \mathbb{P}^1$ and then we have $\mathcal{T}_D \cong \mathcal{O}_{\mathbb{P}^1}(\gamma)$ with $\gamma < 0$. Any vector bundle $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ on \mathbb{P}^1 with $a_1 \geq \dots \geq a_r$ can be rewritten as

$$(7) \quad \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(b_1)^{\oplus r_1} \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(b_s)^{\oplus r_s},$$

with $\sum_{i=1}^s r_i = r$ and $b_1 > \dots > b_s$, i.e. in the Harder-Narasimhan filtration (2) associated to \mathcal{E} , we have $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i}$. Now consider $\mathbb{U}_{\mathbb{P}^1}(s; r_1, d_1; \dots; r_s, d_s)$ with $b_i := d_i/r_i$ and then it is a single point set, consisting only of \mathcal{E} . Set

$$\Delta := \sum_{1 \leq i < j \leq s} \max\{0, \gamma + 1 + b_i - b_j\}.$$

Then we have $h^0(\text{Hom}(\mathcal{E}, \mathcal{E}(\gamma))) = \Delta$. So the parameter space is a well-defined vector space, or its associated projective space if we consider nonzero co-Higgs fields up to scalar multiplication. We have $\Delta > 0$ if and only if $b_1 + \gamma \geq b_s$.

For any $\Phi \in \text{Hom}(\mathcal{E}, \mathcal{E}(\gamma))$ and any positive integer i , let $\Phi^{(i)} : \mathcal{E} \rightarrow \mathcal{E}(i\gamma)$ be the map obtained by iterating i times a shift of Φ . If $b_1 + i\gamma < b_s$ for some i , then we have $\Phi^{(i)} = 0$ and so Φ is a nilpotent logarithmic co-Higgs field. In particular, if $b_1 + 2\gamma < b_s \leq b_1 + \gamma$, then all logarithmic co-Higgs fields are 2-nilpotents and so we have the following.

Proposition 3.3. *For the bundle \mathcal{E} in (7) with $2\gamma \leq b_s - b_1 < \gamma$, the set of its co-Higgs structures is identified with a Δ -dimensional vector space.*

Now the assumption in (5) is simply $b_1 + \gamma \geq b_s$ and let e be the last integer i such that $b_i > \gamma + b_1$. Then we may write

$$(8) \quad \mathcal{E} \cong \mathcal{E}_+ \oplus \mathcal{E}_-, \quad \text{with } \mathcal{E}_+ \cong \bigoplus_{i=1}^e \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i} \text{ and } \mathcal{E}_- \cong \bigoplus_{i=e+1}^r \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i}.$$

It is possible to have $e = 0$ and so \mathcal{E}_+ is trivial. Then we have $H^0(\text{End}(\mathcal{E})(\gamma)) = H^0(\text{Hom}(\mathcal{E}_-, \mathcal{E})(\gamma))$. Thus in case of \mathbb{P}^1 we may rephrase our question in the set-up of holomorphic triples $(\mathcal{E}_1, \mathcal{E}_2, f)$ with $\mathcal{E}_1 = \mathcal{E}_-$, $\mathcal{E}_2 = \mathcal{E}(\gamma)$ and $f : \mathcal{E}_1 \rightarrow \mathcal{E}_2$. Here, \mathcal{E}_1 and \mathcal{E}_2 are related in a sense that \mathcal{E}_1 is a twist of a factor of \mathcal{E}_2 . So our general problem concerning nonzero maps $\Phi : \mathcal{E} \rightarrow \mathcal{E}(\gamma)$ is equivalent to a problem about nonzero maps $\Phi : \mathcal{E}_- \rightarrow \mathcal{E}(\gamma)$.

3.2. Elliptic curves. Let X be an elliptic curve and use the classification of vector bundles on elliptic curves due to M. Atiyah in [1]. We have $\mathcal{T}_D \cong \mathcal{O}_X(-D)$.

Proposition 3.4. *Fix an integer $s \geq 2$ and consider $\mathbb{U} := \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with $d_s/r_s \leq d_1/r_1 + \gamma$.*

- (i) *There exists $[\mathcal{E}] \in \mathbb{U}$ with $\text{Hom}(\mathcal{E}, \mathcal{E}(-D)) \neq 0$.*
- (ii) *If $d_s/r_s = d_1/r_1 + \gamma$, there is $[\mathcal{E}] \in \mathbb{U}$ with $\text{Hom}(\mathcal{E}, \mathcal{E}(-D)) = 0$.*
- (iii) *If $d_s/r_s < d_1/r_1 + \gamma$, then we have $\text{Hom}(\mathcal{E}, \mathcal{E}(-D)) \neq 0$ for all $[\mathcal{E}] \in \mathbb{U}$.*
- (iv) *If e is the maximal integer such that $d_s/r_s \leq d_1/r_1 + e\gamma$, then we have $\Phi^{(e+1)} = 0$ for every $[\mathcal{E}] \in \mathbb{U}$ and $\Phi \in \text{Hom}(\mathcal{E}, \mathcal{E}(-D)) = 0$.*

Proof. Take $[\mathcal{E}] \in \mathbb{U}$ and set $\mathcal{E}_s := \mathcal{F}_s/\mathcal{F}_{s-1}$. In the set-up of part (iii) we have $\mu(\mathcal{E}_s^\vee \otimes \mathcal{F}_1(-D)) > 0$ and so Riemann-Roch gives $h^0(\mathcal{E}_s^\vee \otimes \mathcal{F}_1(-D)) > 0$. Take as Φ the composition of the surjection $\mathcal{E} \rightarrow \mathcal{E}_s$ with a nonzero map $\mathcal{E}_s \rightarrow \mathcal{F}_1(-D)$ and then the inclusion $\mathcal{F}_1(-D) \hookrightarrow \mathcal{E}(-D)$, proving (iii).

Now assume $d_s/r_s = d_1/r_1 + \gamma$. Take as \mathcal{E}_i any semistable bundle with prescribed numeric data so that \mathcal{E}_1 and \mathcal{E}_s are polystable and no factor of $\mathcal{E}_1(-D)$ is isomorphic to a factor of \mathcal{E}_s . Set $\mathcal{E} := \bigoplus_{i=1}^s \mathcal{E}_i$. Due to the slope, we have $\text{Hom}(\mathcal{E}_i, \mathcal{E}_j(-D)) = 0$

if $(i, j) \neq (s, 1)$. Since every nonzero map between stable bundles with the same slope is an isomorphism, we have $\text{Hom}(\mathcal{E}_s, \mathcal{E}_1(-\mathcal{D})) = 0$ and so $\text{Hom}(\mathcal{E}, \mathcal{E}(-\mathcal{D})) = 0$, proving part (ii).

Under the same situation, set $t := \gcd(|d_s|, r_s)$ and write $r_s = at$ and $d_s = bt$. Then each indecomposable factor of \mathcal{E}_s has rank a and degree b , which is also stable. Pick one of these indecomposable factors, say \mathcal{A} . Now from $d_s/r_s = d_1/r_1 + \gamma$, we see that a divides r_1 . Then we have $r_1/a \in \mathbb{Z}$ and it also divides d_1 , say $r_1 = ap$ and $d_1 = qp$. We also see that $\gcd(a, q) = \gcd(a, b) = 1$ and so \mathcal{E}_1 is a polystable bundle whose factors have rank a and degree $q = b - a\gamma$. Let \mathcal{G} be any polystable vector bundle of rank r_1 and degree d_1 with $\mathcal{A} \otimes \mathcal{O}_X(\mathcal{D})$ as one of its factors. Set $\mathcal{F} := \mathcal{G} \oplus (\oplus_{i=2}^s \mathcal{E}_i)$ and then we have $[\mathcal{F}] \in \mathbb{U}$. Since $\text{Hom}(\mathcal{E}_s, \mathcal{G}(-\mathcal{D})) \neq 0$, we have $\text{Hom}(\mathcal{F}, \mathcal{F}(-\mathcal{D})) \neq 0$, proving part (i).

Part (iv) is obvious. \square

Remark 3.5. In parts (i) and (iii) of Proposition 3.4 the proof gives the existence of a nonzero 2-nilpotent co-Higgs field Φ .

3.3. Higher genus case. Assume that X has genus $g \geq 2$. Note that $\gamma \leq 2 - 2g$. For the pairs of integers (r, d) with $r > 0$, denote by $\mathbb{M}_X(r, d)$ the moduli space of the stable vector bundles of rank r on X with degree d . It is known to be a non-empty, smooth and irreducible quasi-projective variety of dimension $r^2(g - 1) + 1$.

Fix a point $p \in X$ and take any exact sequence on X :

$$(9) \quad 0 \rightarrow \mathcal{A} \xrightarrow{u} \mathcal{B} \rightarrow \mathbb{C}_p \rightarrow 0$$

with \mathcal{A} and \mathcal{B} locally free. Note that $\text{rk}(\mathcal{A}) = \text{rk}(\mathcal{B})$ and that $\deg \mathcal{B} = \deg \mathcal{A} + 1$. Then we say that \mathcal{B} is obtained from \mathcal{A} by applying a *positive elementary transformation at p* and that \mathcal{A} is obtained from \mathcal{B} by applying a *negative elementary transformation at p* . For a fixed \mathcal{A} (resp. \mathcal{B}) the set of all extensions (9) is parametrized by a vector space of dimension $\text{rk}(\mathcal{A})$ (resp. $\text{rk}(\mathcal{B})$); since it is an irreducible variety, we may speak about the general positive elementary transformation of \mathcal{A} (resp. a general negative elementary transformation of \mathcal{B}).

Lemma 3.6. *For $(r, d, k) \in \mathbb{Z}^{\oplus 3}$ with $r, k > 0$, fix a general bundle $[\mathcal{A}] \in \mathbb{M}_X(r, d)$. If \mathcal{B} is obtained from \mathcal{A} by applying k positive elementary transformations, then it is stable.*

Proof. Since the statement is trivial for $r = 1$, we may assume $r \geq 2$.

(a) First assume $k = 1$ with the sequence (9) and that \mathcal{B} is not stable so that there exists a subbundle $\mathcal{G}_t \subset \mathcal{B}$ of rank $t \in \{1, \dots, r-1\}$ with $\deg \mathcal{G}_t/t \geq (d+1)/r$. Let $\mathcal{C} \subset \mathcal{A}$ be the saturation of $u^{-1}(\mathcal{G})$ and set $a := \deg \mathcal{C}$. Then we have $a \geq \deg u^{-1}(\mathcal{G}) \geq \deg \mathcal{G} - 1$. Since \mathcal{A} is general, we get by [21, Theorem 3.10] or [7, Theorem 2] that $\mu(\mathcal{A}/\mathcal{C}) - \mu(\mathcal{C}) \geq g - 1$, from which we get

$$\frac{d}{r} - \frac{a}{t} \geq \frac{(r-t)(g-1)}{r}.$$

Using this with $\deg(\mathcal{G}_t) \leq a + 1$, we get

$$\begin{aligned} \mu(\mathcal{B}) - \mu(\mathcal{G}_t) &\geq \frac{d+1}{r} - \frac{a+1}{t} \\ &\geq \frac{t-r+(r-t)(g-1)}{rt} \\ &= \frac{(r-t)(g-2)}{rt} \geq 0, \end{aligned}$$

The equality holds if and only if $g = 2$ and $\deg \mathcal{G}_t = a + 1$. Let \tilde{a} be the maximal degree of a rank t subbundle of \mathcal{A} . For arbitrary t and g , Mukai and Sakai proved in [23] that $td - \tilde{a}r \leq t(r-t)g$, while the quoted results also said that $td - \tilde{a}r \geq t(r-t)(g-1)$. The precise value of \tilde{a} is known by an unpublished result of A. Hirschowitz in [17] and [21, Remark 3.14], which says that $td - \tilde{a}r = t(r-t)(g-1) + \varepsilon$, where ε is the only integer such that $0 \leq \varepsilon < r$ and $\varepsilon + t(r-t)(g-1) \equiv td \pmod{r}$.

Now assume $g = 2$. We conclude unless $\varepsilon = 0$, $a = \tilde{a}$ and $\deg \mathcal{G}_t = a + 1$. In this case we use that we take a general positive elementary transformation of \mathcal{A} . Since $\varepsilon = 0$ and \mathcal{A} is general, \mathcal{A} has only finitely many rank t subbundles of maximal degree $a = \tilde{a}$, say \mathcal{N}_i for $1 \leq i \leq \delta$; see [25] and [30]. The fiber $\mathcal{N}_{i|p}$ of \mathcal{N}_i at p is a t -dimensional linear subspace of the fiber $\mathcal{A}_{|p}$ of \mathcal{A} at p , which is an r -dimensional vector space. The union of these t -dimensional linear subspaces \mathcal{N}_i for $1 \leq i \leq \delta$, is a proper subset of $\mathcal{A}_{|p}$. Thus, for a general positive elementary transformation \mathcal{B} of \mathcal{A} at p , the saturation \mathcal{M}_i of \mathcal{N}_i is just \mathcal{N}_i for all i , i.e. $\deg u^{-1}(\mathcal{M}_i) = \deg \mathcal{M}_i$ for all i , contradicting the assumptions $a = \tilde{a}$ and $\deg \mathcal{G}_t = a + 1$.

(b) Now assume $k \geq 2$. The case $k = 1$ proves that a general positive elementary transformation of a stable bundle is stable. Similarly a general negative transformation of a stable bundle is also stable, and so we may apply the step (a) k times to get the assertion. \square

Theorem 3.7. Fix an integer $s \geq 2$ and consider $\mathbb{U} := \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with $d_s/r_s \leq d_1/r_1 + \gamma$.

- (i) There exists $[\mathcal{E}] \in \mathbb{U}$ with $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D) \neq 0$.
- (ii) If $d_s/r_s \geq d_1/r_1 + \gamma + 1 - g$, there is $[\mathcal{E}] \in \mathbb{U}$ with $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D) = 0$.
- (iii) If $d_s/r_s < d_1/r_1 + \gamma + 1 - g$, then $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D) \neq 0$ for all $[\mathcal{E}] \in \mathbb{U}$.

Proof. Take $[\mathcal{E}] \in \mathbb{U}$ and set $\mathcal{E}_s := \mathcal{F}_s/\mathcal{F}_{s-1}$. Since \mathcal{E}_s and \mathcal{F}_1 are semistable, the bundle $\mathcal{E}_s^\vee \otimes \mathcal{F}_1(-D)$ is also semistable. In the set-up of part (iii) we have $\mu(\mathcal{E}_s^\vee \otimes \mathcal{F}_1 \otimes \mathcal{T}_D) > g - 1$ and so Riemann-Roch gives $h^0(\mathcal{E}_s^\vee \otimes \mathcal{F}_1 \otimes \mathcal{T}_D) > 0$. Take as Φ the composition of the surjection $\mathcal{E} \rightarrow \mathcal{E}_s$ with a nonzero map $\mathcal{E}_s \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_D$ and then the inclusion $\mathcal{F}_1 \otimes \mathcal{T}_D \hookrightarrow \mathcal{E} \otimes \mathcal{T}_D$, proving (iii).

Now assume the set-up of (ii) and pick a general element $(\mathcal{E}_1, \dots, \mathcal{E}_s)$ in

$$\mathbb{M}_X(r_1, d_1) \times \dots \times \mathbb{M}_X(r_s, d_s).$$

In particular, each \mathcal{E}_i is a general stable bundle in $\mathbb{M}_X(r_i, d_i)$. Set $\mathcal{E} := \oplus_{i=1}^s \mathcal{E}_i$ and then it is sufficient to prove the following claim for (ii).

Claim 1: We have $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D) = 0$.

Proof of Claim 1: Since $\mathcal{E} := \oplus_{i=1}^s \mathcal{E}_i$, it is enough to prove that $H^0(\mathcal{E}_i, \mathcal{E}_j \otimes \mathcal{T}_D) = 0$ for all i, j . We have $H^0(\mathcal{E}_i, \mathcal{E}_i \otimes \mathcal{T}_D) = 0$ for each i , because \mathcal{E}_i is stable and $\gamma < 0$. Now assume $i \neq j$. Note that $(\mathcal{E}_i^\vee, \mathcal{E}_j \otimes \mathcal{T}_D)$ is a general element of

$$\mathbb{M}_X(r_i, -d_i) \times \mathbb{M}_X(r_j, d_j + r_j\gamma).$$

We have $\mu(\mathcal{E}_i^\vee \otimes \mathcal{E}_j \otimes \mathcal{T}_D) = -\mu(\mathcal{E}_i) + \mu(\mathcal{E}_j) + \gamma \leq g - 1$. By a theorem of A. Hirschowitz in [28, Theorem 1.2], we have $h^0(\mathcal{E}_i^\vee \otimes \mathcal{E}_j \otimes \mathcal{T}_D) = 0$, concluding the proof of *Claim 1*.

Now we prove part (i). Let \mathcal{B}_i be a semistable bundle of rank r_i on X with degree d_i for each i , and let $\mathcal{E} := \oplus_{i=1}^s \mathcal{B}_i$. Our strategy is to find appropriate \mathcal{B}_1 and \mathcal{B}_s with the additional condition $\text{Hom}(\mathcal{B}_s, \mathcal{B}_1 \otimes \mathcal{T}_D) \neq 0$, which would imply part (i).

(a) Assume $r_s < r_1$. Setting $r' := r_1 - r_s$ and $d' := d_1 + \gamma r_1 - d_s$, it is enough to show the existence of an exact sequence of vector bundles on X :

$$(10) \quad 0 \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \rightarrow 0,$$

with $\mathcal{A}_1, \mathcal{A}_2$ semistable and \mathcal{A}_1 of rank r_s and degree d_s , \mathcal{A}_2 of rank r_1 and degree $d_1 + \gamma r_1$. Then \mathcal{A}_3 would be of rank r' and degree d' , and we may take $\mathcal{B}_s := \mathcal{A}_1$ and $\mathcal{B}_1 := \mathcal{A}_2 \otimes \mathcal{T}_D^\vee$.

Note that for a quadruple of integers $(x_1, x_2, a_1, a_2) \in \mathbb{Z}^{\oplus 4}$ with $x_2 > x_1 > 0$ and $a_1/x_1 \leq a_2/x_2$ (resp. $a_1/x_1 < a_2/x_2$), we have

$$\frac{a_2}{x_2} \leq \frac{a_2 - a_1}{x_2 - x_1} \quad \left(\text{resp.} \quad \frac{a_2}{x_2} < \frac{a_2 - a_1}{x_2 - x_1} \right).$$

Using the above to $(x_1, x_2, a_1, a_2) = (r_s, r_1, d_s, d_1 + \gamma r_1)$, together with

$$\mu_-(\mathcal{E}) = d_s/r_s \leq d_1/r_1 + \gamma = \mu_-(\mathcal{E} \otimes \mathcal{T}_D),$$

we have $d_1/r_1 + \gamma \leq d'/r'$ with equality if and only if $d_s/r_s = d_1/r_1 + \gamma$. When the equality holds, we may take as \mathcal{A}_1 and \mathcal{A}_3 arbitrary semistable bundles with the prescribed ranks and degrees and then take $\mathcal{A}_2 := \mathcal{A}_1 \oplus \mathcal{A}_3$. Now assume $d_s/r_s < d_1/r_1 + \gamma$ and so $d_1/r_1 + \gamma < d'/r'$. In this case by the positive answer to the conjecture of Lange, there is an exact sequence (10) of vector bundles on X with the prescribed ranks and degrees and with stable $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 ; see [28, Introduction].

(b) Assume $r_s > r_1$. Similarly as in (a) we set $r'' := r_s - r_1$ and $d'' := d_1 - \gamma r_1 - d_s$. By taking $\mathcal{B}_s := \mathcal{A}_2$ and $\mathcal{B}_1 := \mathcal{A}_3 \otimes \mathcal{T}_D^\vee$, it is sufficient to find an exact sequence (10) with \mathcal{A}_2 and \mathcal{A}_3 semistables, \mathcal{A}_1 of rank r'' and degree d'' , \mathcal{A}_2 of rank r_s and degree d_s and \mathcal{A}_3 of rank r_1 and degree $d_1 + \gamma r_1$.

First assume $d_s/r_s = d_1/r_1 + \gamma$. In this case we have $d''/r'' = d_s/r_s$ and we take as \mathcal{A}_1 and \mathcal{A}_3 arbitrary semistable bundles with prescribed ranks and degrees and set $\mathcal{A}_2 := \mathcal{A}_1 \oplus \mathcal{A}_3$. Now assume $d_s/r_s < d_1/r_1 + \gamma$ and so $d_1/r_1 + \gamma > d''/r''$. Again by the conjecture of Lange proved in [28] we may take $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ with the prescribed ranks and degree and stable.

(c) Assume $r_s = r_1$. First assume $d_s/r_s = d_1/r_1 + \gamma$, i.e. $d_1 = d_s - \gamma r_1$. In this case we take as \mathcal{B}_s any semistable bundle with rank r_s and degree d_s and set $\mathcal{B}_1 := \mathcal{B}_s \otimes \mathcal{T}_D$. Now assume $k := d_1 + r_1 \gamma - d_s > 0$. We take as \mathcal{B}_s a general stable bundle of rank r_s and degree d_s . Then $\mathcal{B}_s \otimes \mathcal{T}_D$ is a general element of $\mathbb{M}_X(r_1, d_1 - t)$. We take as \mathcal{B}_1 a bundle obtained from $\mathcal{B}_s \otimes \mathcal{T}_D$ by applying k general positive elementary transformations. \square

Remark 3.8. Consider a smooth algebraic curve X of an arbitrary genus $g \geq 0$ and assume $s \geq 3$ together with

$$d_2/r_2 + \gamma < d_s/r_s \leq d_1/r_1 + \gamma.$$

By Theorem 3.7 in case $g \geq 2$ and Proposition 3.4 for $g = 1$, there is (\mathcal{E}, Φ) with $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ and a nonzero map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_D$. Take any $[\mathcal{E}] \in$

$\mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with the Harder-Narasimhan filtration (2). Note that we have $\text{Hom}(\mathcal{A}, \mathcal{B}) = 0$ for any semistable bundles \mathcal{A} and \mathcal{B} with $\mu(\mathcal{A}) > \mu(\mathcal{B})$. *Claim 1* in the proof of Theorem 3.7 applied to $\mathcal{E}/\mathcal{F}_1$ shows that any map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ is uniquely determined by $f : \mathcal{E}/\mathcal{F}_{s-1} \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$; moreover we get $\text{Im}(\Phi) = \text{Im}(f)$ and $\ker(\Phi)$ is the inverse image of $\ker(f)$ under the surjection $\mathcal{E} \rightarrow \mathcal{E}/\mathcal{F}_{s-1}$.

Conversely, for $1 \leq i \leq s$, choose arbitrary semistable bundles \mathcal{E}_i with $\text{rk}(\mathcal{E}_i) = r_i$ and $\deg \mathcal{E}_i = d_i$, and a map $f : \mathcal{E}_s \rightarrow \mathcal{E}_1 \otimes \mathcal{T}_{\mathcal{D}}$. To get a vector bundle $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$, we only need to consider $(s-1)$ extension classes

$$0 \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_{i+1} \rightarrow \mathcal{E}_{i+1} \rightarrow 0$$

for $i = 1, \dots, s-1$, where $\mathcal{F}_1 := \mathcal{E}_1$. Once \mathcal{E} is chosen, the map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ is uniquely determined by f .

In case of curves, we sometimes may improve Remark 2.3 to a strict inequality in the following way.

Remark 3.9. Take $[\mathcal{E}] \in \mathbb{U} := \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with the Harder-Narasimhan filtration (2). Assume that $s \geq 2$ with $r_s \neq r_1$,

$$\gcd(r_1, d_1) = \gcd(r_s, d_s) = 1 \text{ and } d_1/r_1 + \gamma \geq d_s/r_s.$$

Since $\gcd(r_1, d_1) = 1$, the sheaf \mathcal{F}_1 is stable and so $\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ is stable. Since $\gcd(r_s, d_s) = 1$, the sheaf $\mathcal{F}_s/\mathcal{F}_{s-1}$ is also stable. From $r_1 \neq r_s$ we get that $\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ and $\mathcal{F}_s/\mathcal{F}_{s-1}$ are not isomorphic and so we have $\text{Hom}(\mathcal{F}_s/\mathcal{F}_{s-1}, \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}) = 0$. If $s \geq 3$, we obviously have $\text{Hom}(\mathcal{F}_i/\mathcal{F}_{i-1}, \mathcal{F}_j \otimes \mathcal{T}_{\mathcal{D}}) = 0$ for all $i, j \in \{1, \dots, s\}$ with $(i, j) \neq (s, 1)$, even without the assumptions $r_s \neq r_1$ and $\gcd(r_1, d_1) = \gcd(r_s, d_s) = 1$. Thus we have $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}) = 0$.

Remark 3.10. In the following exact sequence

$$(11) \quad 0 \rightarrow \mathcal{A} \rightarrow \mathcal{C} \rightarrow \mathcal{B} \rightarrow 0$$

of vector bundles on X with \mathcal{A} and \mathcal{B} semistable, if we have $\mu(\mathcal{A}) + 2 - 2g > \mu(\mathcal{B})$, then we have $h^1(\mathcal{A} \otimes \mathcal{B}^\vee) = 0$ and so (11) splits. Thus if we have

$$\frac{d_i}{r_i} + 2 - 2g > \frac{d_{i+1}}{r_{i+1}}$$

for all i , then we have $\mathcal{E} \cong \text{gr}(\mathcal{E})$ for all $[\mathcal{E}] \in \mathbb{U} := \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$.

Now assume $d_i/r_i + 2 - 2g \geq d_{i+1}/r_{i+1}$ for all i . If the equality holds for some i , i.e. $d_i/r_i = d_{i+1}/r_{i+1} + 2g - 2$, then we have $r_i \neq r_{i+1}$ and that r_h and d_h are coprime, where $h \in \{i, i+1\}$ is the index with higher rank $r_h = \max\{r_i, r_{i+1}\}$. As in Remark 3.9 we get $\mathcal{E} \cong \text{gr}(\mathcal{E})$ for all $[\mathcal{E}] \in \mathbb{U}$.

For example, take $s = 2$. We just proved that $\mathcal{E} \cong \mathcal{F}_1 \oplus \mathcal{F}_2/\mathcal{F}_1$ for all $[\mathcal{E}] \in \mathbb{U}(2; 1, d_1; 1, d_2)$ with a nonzero map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ and either $\mathcal{D} \neq 0$ or $r_1 \neq r_2$, and $d_h, r_h \in \mathbb{Z}$, where $r_h = \max\{r_1, r_2\}$.

Example 3.11. Assume $d_1 > d_2 - \gamma$. For a fixed $\mathcal{R} \in \text{Pic}^{d_2}(X)$, consider the set

$$\mathbb{E} := \{(\mathcal{F}_1, \psi) \mid \mathcal{F}_1 \in \text{Pic}^{d_1}(X) \text{ and } 0 \neq \psi : \mathcal{R} \rightarrow \mathcal{F}_1 \times \mathcal{T}_{\mathcal{D}}\} / \sim,$$

where the equivalent relation \sim is given by $(\mathcal{F}_1, \psi) \sim (\mathcal{F}_1, c\psi)$ for all $c \in \mathbb{C}^*$. \mathbb{E} is the set of all effective divisors of X with degree $d_1 + \gamma - d_2$ and so \mathbb{E} is isomorphic to a symmetric product of $d_1 + \gamma - d_2$ copies of X and in particular it is irreducible. By Remark 3.10 we have $\mathcal{E} \cong \mathcal{F}_1 \oplus \mathcal{F}_2/\mathcal{F}_1$ for all $[\mathcal{E}] \in \mathbb{U}_X(2; 1, d_2 + \gamma; 1, d_2)$.

Example 3.12. Assume $g \geq 2$ and take $\mathcal{D} = \emptyset$ so that $\gamma = 2 - 2g$. Fix any $d \in \mathbb{Z}$ and consider $\mathcal{E} \in \mathbb{U}_X(2; 1, d + 2g - 2; 1, d)$ with a nonzero map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes T_X$. Set $\mathcal{R} := \mathcal{F}_2/\mathcal{F}_1 \in \text{Pic}^d(X)$ and then Φ is induced by a nonzero map $\psi : \mathcal{R} \rightarrow \mathcal{F}_1 \otimes T_X$. Since \mathcal{F}_1 is in $\text{Pic}^{d+2g-2}(X)$, the map ψ is an isomorphism. Thus we get $\mathcal{F}_1 \cong \mathcal{R} \otimes \omega_X$ and that for a fixed \mathcal{E} the set of all nonzero map Φ is parametrized by a nonzero scalar. From $h^1(\omega_X) = 1$ we see that there are, up to isomorphism, exactly two vector bundles \mathcal{E} fitting into an exact sequence

$$(12) \quad 0 \rightarrow \mathcal{R} \otimes \omega_X \rightarrow \mathcal{E} \rightarrow \mathcal{R} \rightarrow 0,$$

that is, $(\mathcal{R} \otimes \omega_X) \oplus \mathcal{R}$ and an indecomposable bundle. Thus the set of all (\mathcal{E}, Φ) , up to isomorphisms, with nonzero Φ , is parametrized one-to-one by the disjoint union of two copies of $\text{Pic}^d(X) \times \mathbb{C}^*$. Thus no one-to-one parameter space is irreducible. We get another irreducible parameter space that is not one-to-one, by taking as parameter space, up to a nonzero constant, the relative Ext^1 group of (12) over $\text{Pic}^d(X)$; each indecomposable bundle \mathcal{E} appears ∞^1 -times and it has $gr(\mathcal{E}) \cong (\mathcal{R} \otimes \omega_X) \oplus \mathcal{R}$ as its limit inside the parameter space.

Now for s at least two let us define the set $\mathbb{U}^{\text{co}} = \mathbb{U}_X^{\text{co}}(s; r_1, d_1; \dots; r_s, d_s)$ of certain co-Higgs bundles associated to $\mathbb{U} = \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ as follows.

$$\mathbb{U}^{\text{co}} := \{(\mathcal{E}, \Phi) \mid [\mathcal{E}] \in \mathbb{U} \text{ and } \Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}} \text{ with } \text{Im}(\Phi) \subseteq \mathcal{F}_1 \text{ and } \ker(\Phi) \subseteq \mathcal{F}_{s-1} \\ \text{such that } \text{rk}(\text{Im}(\Phi)) = \min\{r_1, r_s\} \text{ and } \mathcal{F}_1, \mathcal{F}_s/\mathcal{F}_{s-1} \text{ stable}\}$$

Denote by $\Gamma \subseteq \mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$ the set of all pairs $(\mathcal{F}_1, \mathcal{F}_s/\mathcal{F}_{s-1})$ obtained from \mathbb{U}^{co} and call the projection from Γ to each factor by π_1 and π_2 , respectively.

Proposition 3.13. *Assume that*

- *each d_i is positive such that $d_i/r_i > d_{i+1}/r_{i+1}$ for all i , and*
- *$d_1/r_1 + \gamma \geq d_s/r_s$.*

Then we have the following assertions.

- (i) *If $r_1 = r_s$, then π_1 and π_2 are dominant.*
- (ii) *If $r_1 < r_s$ (resp. $r_1 > r_s$) and $d_1/r_1 + \gamma > d_s/r_s$, then π_1 (resp. π_2) is dominant.*
- (iii) *Assume $d_1/r_1 + \gamma > g - 1 + d_s/r_s$. Then Γ contains a non-empty open subset of $\mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$; if $r_1 \geq r_s$, then we have $\ker(\Phi) = \mathcal{F}_{s-1}$.*

Proof. Fix a point $([\mathcal{A}_1], [\mathcal{A}_s]) \in \mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$. For arbitrary $[\mathcal{A}_i] \in \mathbb{M}_X(r_i, d_i)$, $i = 2, \dots, s-1$, we consider $\mathcal{E} := \bigoplus_{i=1}^s \mathcal{A}_i$ with the Harder-Narasimhan filtration (2) such that $\mathcal{F}_1 \cong \mathcal{A}_1$ and $\mathcal{F}_s/\mathcal{F}_{s-1} \cong \mathcal{A}_s$. We take a map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with $\Phi(\mathcal{F}_{s-1}) = 0$, which is induced by a map $\psi : \mathcal{A}_s \rightarrow \mathcal{A}_1 \otimes \mathcal{T}_{\mathcal{D}}$ whose existence is guaranteed by the assumptions.

First assume $r_s = r_1$. We need to prove the existence of a map ψ of rank r_1 when \mathcal{A}_1 is general in $\mathbb{M}_X(r_1, d_1)$ and \mathcal{A}_s is general in $\mathbb{M}_X(r_s, d_s)$; we do not claim here that $([\mathcal{A}_1], [\mathcal{A}_2])$ is general in $\mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$. The dominance of π_2 is the content of Lemma 3.6. The dominance of π_1 can be proved by applying the dual map, or with the same proof as in the proof of Lemma 3.6, concluding part (i).

Now assume $r_s < r_1$ and $d_1/r_1 + \gamma > d_s/r_s$. We take as \mathcal{A}_s a general element of $\mathbb{M}_X(r_s, d_s)$. The existence of a stable $\mathcal{A}_1 \in \mathbb{M}_X(r_1, d_1)$ with an embedding $\psi : \mathcal{A}_s \hookrightarrow \mathcal{A}_1 \otimes \mathcal{T}_{\mathcal{D}}$ with $\mathcal{A}_1/\psi(\mathcal{A}_s)$ stable and general in $\mathbb{M}_X(r_s - r_1, d_1 - \gamma r_1)$ is proved in part (i) of the proof of Theorem 3.7.

By using part (ii) of the proof of Theorem 3.7 instead of part (i), we get the case $r_s > r_1$ and $d_1/r_1 + \gamma > d_s/r_s$.

Now consider part (iii) and assume $d_1/r_1 + \gamma > g - 1 + d_s/r_s$. Take a general $([\mathcal{A}_1], [\mathcal{A}_s]) \in \mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$ and set $\mathcal{B}_1 := \mathcal{A}_1 \otimes \mathcal{T}_{\mathcal{D}}$. Then it is sufficient to find $\psi : \mathcal{A}_s \rightarrow \mathcal{B}_1$ with $\text{Im}(\psi) = \min\{r_1, r_s\}$. By the assumptions, we have $\mu(\mathcal{A}_s^\vee \otimes \mathcal{B}_1) > g - 1$ and so Riemann-Roch gives $\text{Hom}(\mathcal{A}_s, \mathcal{B}_1) \neq 0$. Take a general element $\psi \in \text{Hom}(\mathcal{A}_s, \mathcal{B}_1)$ and then it is sufficient to prove that ψ has rank $\min\{r_1, r_s\}$. Note that we have $h^1(\mathcal{A}_s^\vee \otimes \mathcal{B}_1) = 0$ and so \mathcal{B}_1 is an element of the following set

$$\mathbb{W} := \{[\mathcal{F}] \in \mathbb{M}_X(r_1, d_1 + \gamma r_1) \mid h^1(\mathcal{A}_s^\vee \otimes \mathcal{F}) = 0\}.$$

By Riemann-Roch, we have the following, for each $[\mathcal{F}] \in \mathbb{W}$,

$$h^0(\mathcal{A}_s^\vee \otimes \mathcal{F}) = \deg(\mathcal{A}_s^\vee \otimes \mathcal{F}) + r_1 r_2 (1 - g) = r_1 r_2 \left(\frac{d_1}{r_1} + \gamma - \frac{d_s}{r_s} + 1 - g \right) > 0.$$

Now take the relative Hom with \mathbb{W} as its parameter space, i.e. for each $[\mathcal{F}] \in \mathbb{W}$, the fibre is $\text{Hom}(\mathcal{A}_s, \mathcal{F})$. The total space Λ of this relative Hom is irreducible, because $h^0(\mathcal{A}_s^\vee \otimes \mathcal{F})$ is constant for all $[\mathcal{F}] \in \mathbb{W}$ by [6] and [20]. By [5, part (d) of Theorem 1.2], a general element $(\varphi : \mathcal{A}_s \rightarrow \mathcal{F})$ of Λ has φ with rank $\min\{r_1, r_s\}$. When $r_1 \geq r_s$, this implies that $\ker(\Phi) = \mathcal{F}_{s-1}$, because the map $\psi : \mathcal{F}_s/\mathcal{F}_{s-1} \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ is injective if and only if it has rank r_s . \square

Remark 3.14. As in the end of proof of Proposition 3.13, to show that the set of the co-Higgs bundles (\mathcal{E}, Φ) with certain properties is parametrized by an irreducible variety, it sometimes works to prove that (a) the set of all bundles \mathcal{E} is parametrized by an irreducible variety Y , and (b) the integer $k := \dim \text{Hom}(\mathcal{E}_y, \mathcal{E}_y \otimes \mathcal{T}_{\mathcal{D}})$ is constant for all $y \in Y$. In this case, the set of all (\mathcal{E}, Φ) with no restriction on Φ is parametrized by a vector bundle of rank k on Y .

Example 3.15. Assume $r_1 = r_s$ and $d_1/r_1 + \gamma = d_s/r_s$. Consider a bundle $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with an arbitrary map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. Since $d_i/r_i > d_1/r_1 + \gamma$ for all $i < s$, there is no nonzero map $\mathcal{F}_i/\mathcal{F}_{i-1} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ and so we have $\mathcal{F}_{s-1} \subseteq \ker(\Phi)$. On the other hand, since $d_s/r_s > d_j/r_j + \gamma$ for all $j > 1$, we have $\Phi(\mathcal{E}) \subseteq \mathcal{F}_1$. We have $\text{rk}(\Phi) = r_1$ if and only if $\mathcal{F}_s/\mathcal{F}_{s-1} \cong \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ and Φ is induced by an isomorphism $\mathcal{F}_s/\mathcal{F}_{s-1} \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$.

Example 3.16. Assume $r_1 = r_s$ and that

$$d_2/r_2 + \gamma < d_s/r_s \text{ and } d_{s-1}/r_{s-1} > d_1/r_1 + \gamma.$$

If we choose (\mathcal{E}, Φ) with $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$, then as in Example 3.15 we see that $\Phi(\mathcal{E}) \subseteq \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ and $\mathcal{F}_{s-1} \subseteq \ker(\Phi)$, because $d_i/r_i + \gamma < d_s/r_s$ for all $i > 1$ and $d_j/r_j > d_1/r_1 + \gamma$ for all $j < s$. Set $k := d_1 - d_2 + \gamma r_1$. Then we have $\text{rk}(\Phi) = r_1$ if and only if $\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ is obtained from $\mathcal{F}_s/\mathcal{F}_{s-1}$ by applying k positive elementary transformations and Φ is induced by the associated inclusion $\mathcal{F}_s/\mathcal{F}_{s-1} \hookrightarrow \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$.

4. SEGRE INVARIANT

In this section, we do not assume that $\mathcal{T}_{\mathcal{D}}$ has some kind of negativity, so that we may have stable (\mathcal{E}, Φ) with nonzero Φ . Let \mathcal{E} be a torsion-free sheaf of rank $r \geq 2$ and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ a co-Higgs field. For a fixed integer $k \in \{1, \dots, r-1\}$, let us denote by $\mathcal{S}(k, \mathcal{E}, \Phi)$ the set of all subsheaves $\mathcal{A} \subset \mathcal{E}$ of rank k such that $\Phi(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{T}_{\mathcal{D}}$. Define the k^{th} -Segre invariant to be

$$s_k(\mathcal{E}, \Phi) := k \deg \mathcal{E} - \max_{\mathcal{A} \in \mathcal{S}(k, \mathcal{E}, \Phi)} r \deg \mathcal{A}.$$

In case $\Phi = 0$, we simply denote it by $s_k(\mathcal{E})$. This is an extension of the Segre invariant, introduced in [19] with the notation $s_k(\mathcal{E})$, to the case $n \geq 2$. Over curves this notion was used in several literatures, including [5, 7, 8, 17, 21, 25, 28, 30]. If we take \mathcal{T}_D^\vee instead of \mathcal{T}_D , we get a definition for logarithmic Higgs fields. Note that we always have $\mathcal{S}(k, \mathcal{E}, 0) \neq \emptyset$ and $s_k(\mathcal{E}) \leq s_k(\mathcal{E}, \Phi)$.

Lemma 4.1. *Let (\mathcal{E}, Φ) be a 2-nilpotent co-Higgs bundle of rank r , and set $\mathcal{A} := \ker(\Phi)$ and $\mathcal{B} := \text{Im}(\Phi)$ with $r' := \text{rk}(\mathcal{A})$. Then we have the following:*

- (i) $\mathcal{A} \in \mathcal{S}(r', \mathcal{E}, \Phi)$;
- (ii) $\mathcal{S}(k, \mathcal{A}, 0) \subseteq \mathcal{S}(k, \mathcal{E}, \Phi)$ for $1 \leq k < r'$;
- (iii) \mathcal{B} is torsion-free and $\Phi^{-1}(\mathcal{G}) \in \mathcal{S}(k, \mathcal{E}, \Phi)$ for all $\mathcal{G} \in \mathcal{S}(k - r', \mathcal{B}, 0)$ and $r' < k < r$;
- (iv) $\mathcal{S}(k, \mathcal{E}, \Phi) \neq \emptyset$ for all k .

Proof. Parts (i) and (ii) are obvious. Part (iii) is true, because $\mathcal{B} \subseteq \mathcal{A} \otimes \mathcal{T}_D$ by the definition of 2-nilpotent. Part (iv) follows from the other ones. \square

Example 4.2. From [3, Theorem 1.1] we get a description of the set of nilpotent co-Higgs structures on a fixed stable bundle of rank two on \mathbb{P}^n . Indeed it is either trivial or an $(n + 1)$ -dimensional vector space, depending on the parity of the first Chern class and an invariant $x_{\mathcal{E}}$. We get a non-trivial set of nilpotent co-Higgs structures on \mathcal{E} if and only if $c_1(\mathcal{E}) + 2x_{\mathcal{E}} = -3$, and in this case we get $s_1(\mathcal{E}, \Phi) = 1$.

4.1. Curve case. From now on we assume $n = 1$ with $g = g(X)$ and $\gamma < 0$. Take (\mathcal{E}, Φ) with $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ and let (2) be the Harder-Narasimhan filtration of \mathcal{E} .

Remark 4.3. From the assumption $\gamma < 0$, we have $\Phi(\mathcal{F}_i) \subseteq \mathcal{F}_{i-1} \otimes \mathcal{T}_D$.

Remark 4.3 immediately proves the following two lemmas.

Lemma 4.4. *For an integer $j \in \{1, \dots, s - 1\}$, set $k(j) = \sum_{i=1}^j r_i$. Then*

$$s_{k(j)}(\mathcal{E}, \Phi) \leq k(j) \deg \mathcal{E} - r \deg \mathcal{F}_j$$

Remark 4.5. We expect that the inequality in Lemma 4.4 is in fact equality, although we give the positive answers only to some special cases; see Lemma 4.6 and Proposition 4.10.

Lemma 4.6. *Assume $g = 0$ and take $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq a_j$ for all $i \leq j$. Then we have*

$$s_k(\mathcal{E}, \Phi) = s_k(\mathcal{E}) = k(a_1 + \dots + a_r) - r(a_1 + \dots + a_k).$$

Proposition 4.7. *Fix $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with $s \geq 2$ and $\Phi \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D)$. Choose any $k \in \{r_1 + 1, \dots, r - r_s + 1\}$ such that there is $h \in \{1, \dots, s - 1\}$ with $r_1 + \dots + r_h < k < r_1 + \dots + r_{h+1}$, and set*

$$\begin{aligned} r' &:= k - r_1 - \dots - r_s, \\ e &:= s_k(\mathcal{F}_{h+1}/\mathcal{F}_h), \\ d' &:= r' \deg \mathcal{F}_{h+1}/\mathcal{F}_h - e r_{h+1}. \end{aligned}$$

Let $\mathcal{B} \subset \mathcal{F}_{h+1}/\mathcal{F}_h$ be any subsheaf of rank r' and degree d' , and set $\mathcal{A} := u^{-1}(\mathcal{B})$, where u is the surjection in the exact sequence

$$0 \rightarrow \mathcal{F}_h \rightarrow \mathcal{F}_{h+1} \xrightarrow{u} \mathcal{F}_{h+1}/\mathcal{F}_h \rightarrow 0.$$

Then $\mathcal{B} \in \mathcal{S}(k, \mathcal{E}, \Phi)$ and $s_k(\mathcal{E}, \Phi) \leq k \deg \mathcal{E} - k(\deg \mathcal{F}_h + e)$.

Proof. Note that d' is the degree of all rank r' maximal degree subsheaves of $\mathcal{F}_{h+1}/\mathcal{F}_h$. Since $\deg \mathcal{A} = d'$, we have $\deg \mathcal{B} = \deg \mathcal{F}_h + d'$. Since $\Phi(\mathcal{F}_{h+1}) \subset \mathcal{F}_h \subset \mathcal{B}$, we have $\mathcal{B} \in \mathcal{S}(k, \mathcal{E}, \Phi)$. Since $\deg \mathcal{B} = \deg \mathcal{F}_h + \deg \mathcal{A}$, we get the assertion. \square

Now Lemma 4.6 and Proposition 4.7 prove the following result.

Corollary 4.8. *The Segre invariant $s_k(\mathcal{E}, \Phi)$ is defined for all (\mathcal{E}, Φ) , if $\gamma < 0$.*

Example 4.9 shows that in Proposition 4.7 we may have strict inequality; of course, to be in the set-up of Proposition 4.7 we need to have $r_{h+1} \geq 2$.

Example 4.9. Assume $g \geq 5$ and fix $h \in \{1, \dots, s-2\}$ with $s \geq 3$. Set $r_{h+1} = r_{h+2} = 2$ and fix $r_i > 0$ for $i \notin \{h+1, h+2\}$ and $d_i \in \mathbb{Z}$, $i = 1, \dots, s$ such that

- $d_i/r_i > d_{i+1}/r_{i+1}$ for all $i = 1, \dots, s-1$ and
- $d_{h+1} = 2d_{h+2} + 1$.

By a theorem of Nagata there is a stable bundle \mathcal{E}_{h+1} of rank 2 with degree d_{h+1} and $g-1 \leq s_1(\mathcal{E}_{h+1}) \leq g$. Here, $s_1(\mathcal{E}_{h+1})$ is the only integer t with $g-1 \leq t \leq g$ and $d_{h+1}-t$ even. For $i \neq h+1$ we choose \mathcal{E}_i to be any semistable bundle of degree d_i and rank r_i . Set $\mathcal{E} := \bigoplus_{i=1}^s \mathcal{E}_i$ and then we have $\mathcal{F}_i = \bigoplus_{j=1}^i \mathcal{E}_j$ in the Harder-Narasimhan filtration (2) of \mathcal{E} .

Take any $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with $\ker(\Phi) \supseteq \bigoplus_{i=0}^{h+1} \mathcal{E}_i$, e.g. take $\Phi = 0$ or, for certain \mathcal{E}_1 and \mathcal{E}_s so that there is a nonzero map $\mathcal{E}_s \rightarrow \mathcal{E}_1 \otimes \mathcal{T}_{\mathcal{D}}$, take a 2-nilpotent map Φ with $\ker(\Phi) \supseteq \mathcal{F}_{s-1}$. Let $\mathcal{A} \subset \mathcal{F}_{h+1}/\mathcal{F}_h$ be a line subbundle of maximal degree and then we have $\mathcal{B} := u^{-1}(\mathcal{A}) = (\sum_{i=1}^h \mathcal{E}_i) \oplus \mathcal{A}$. If we set $\mathcal{B}_1 := (\sum_{i=1}^h \mathcal{E}_i) \oplus \mathcal{E}_{h+2}$, then we have $\deg \mathcal{B}_1 > \deg \mathcal{B}$.

Proposition 4.10. *Assume $r_i = 1$ for all i . For an integer $k \in \{1, \dots, r-1\}$ and any co-Higgs bundle (\mathcal{E}, Φ) with $[\mathcal{E}] \in \mathbb{U}_X(r; 1, d_1; \dots; 1, d_r)$, we have*

$$s_k(\mathcal{E}) = s_k(\mathcal{E}, \Phi) = k \deg \mathcal{E} - r \deg \mathcal{F}_k$$

and \mathcal{F}_k is the only bundle achieving the minimum degree in $\mathcal{S}(k, \mathcal{E}, \Phi)$.

Proof. By Remark 4.3 and Lemma 4.4, we have $[\mathcal{F}_k] \in \mathcal{S}(k, \mathcal{E}, \Phi)$. Thus it is sufficient to prove that \mathcal{F}_k is the only one achieving the minimum degree in $\mathcal{S}(k, \mathcal{E}, 0)$. Take any $[\mathcal{G}] \in \mathcal{S}(k, \mathcal{E}, 0)$ with maximal degree. The maximality condition on $\deg \mathcal{G}$ implies that \mathcal{E}/\mathcal{G} has no torsion and so it is a vector bundle of rank $r-k$ on X . We use double induction on k and r . The case $k = 1$ is obvious, because \mathcal{F}_1 is the first step of the Harder-Narasimhan filtration of \mathcal{E} .

Assume that k is at least two and the proposition holds for trivial co-Higgs fields with any $k' \in \{1, \dots, k-1\}$ and any bundles \mathcal{E}' whose Harder-Narasimhan filtration has rank one bundles as subquotients.

Assume for the moment $\mathcal{F}_1 \subset \mathcal{G}$. Since \mathcal{F}_1 is saturated in \mathcal{E} , i.e. $\mathcal{E}/\mathcal{F}_1$ has no torsion, \mathcal{F}_1 is saturated in \mathcal{G} and $\mathcal{G}/\mathcal{F}_1$ is a rank $k-1$ subsheaf of the vector bundle $[\mathcal{E}/\mathcal{F}_1] \in \mathbb{U}_X(r-1; 1, d_2; \dots; 1, d_r)$. The inductive assumption gives $\deg \mathcal{G}/\mathcal{F}_1 \leq \deg \mathcal{F}_k/\mathcal{F}_1$, with equality if and only if $\mathcal{G}/\mathcal{F}_1 \cong \mathcal{F}_k/\mathcal{F}_1$, i.e. $\deg \mathcal{G} \leq \deg \mathcal{F}_k$ with equality if and only if $\mathcal{G} \cong \mathcal{F}_k$.

Now assume $\mathcal{F}_1 \not\subset \mathcal{G}$. Since \mathcal{G} is saturated in \mathcal{E} , this means that $\mathcal{F}_1 + \mathcal{G}$ has rank $k+1$. Let \mathcal{N} be the saturation of $\mathcal{F}_1 + \mathcal{G}$ in \mathcal{E} , and then we have $\deg \mathcal{N} \geq \deg \mathcal{F}_1 + \deg \mathcal{G}$ and $\mathcal{N}/\mathcal{F}_1$ is a rank k subsheaf of $\mathcal{E}/\mathcal{F}_1$. If $r \geq k-2$, then by the inductive assumption on r we have $\deg \mathcal{N}/\mathcal{F}_1 \leq \deg \mathcal{F}_{k+1}/\mathcal{F}_1 < \deg \mathcal{F}_k - \deg \mathcal{F}_1$ and so $\deg \mathcal{G} < \deg \mathcal{F}_k$, a contradiction. Thus we may assume $k = r-1$ and so $\mathcal{N} \cong \mathcal{E}$. Since $\mathcal{F}_1 + \mathcal{G}$

has rank $k + 1$, the natural map $\mathcal{G} \rightarrow \mathcal{E}/\mathcal{F}_1$ is injective. Thus we have $\deg \mathcal{G} \leq \deg \mathcal{E} - \deg \mathcal{F}_1 < \deg \mathcal{F}_{r-1}$, a contradiction. \square

Now for $k \in \{1, \dots, r-1\}$ set

$$\delta_k(\mathcal{E}, \Phi) := \max_{\mathcal{A} \in \mathcal{S}(k, \mathcal{E}, \Phi)} \deg(\mathcal{A}),$$

$\delta_0(\mathcal{E}, \Phi) := 0$ and $\delta_r(\mathcal{E}, \Phi) := \deg(\mathcal{E})$. In case $\Phi = 0$, we simply denote it by $\delta_k(\mathcal{E})$.

Proposition 4.11. *Fix $h \in \{1, \dots, s\}$ with $s \geq 2$ and set $\rho := r_1 + \dots + r_h$. For $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$, we have*

- (i) $s_\rho(\mathcal{E}) = \rho \deg \mathcal{E} - k \deg \mathcal{F}_h$ and $\mathcal{F}_h \subset \mathcal{E}$ is the only subsheaf of rank ρ with degree $\deg \mathcal{F}_h$;
- (ii) $\deg \mathcal{E} \leq \deg \mathcal{F}_{h-1} + (k - \rho)d_h/r_h$ for k with $\rho - r_h < k < \rho$ and $[\mathcal{G}] \in \mathcal{S}(k, \mathcal{E}, 0)$;
- (iii) $\delta_k(\mathcal{E}) - \deg(\mathcal{F}_{h-1})$ for k with $\rho - r_h < k < \rho$, equals

$$\max \left\{ \sum_{j=h}^s \delta_{t_j}(\mathcal{F}_j/\mathcal{F}_{j-1}) \mid t_h + \dots + t_s = k + r_h - \rho \text{ with } 0 \leq t_j \leq r_j \text{ for all } j \right\}.$$

Proof. Set $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) = d_i/r_i$ for $i = 1, \dots, s$ and let $\mathcal{G} \subseteq \mathcal{E}$ be a rank ρ subsheaf of maximal degree. Then part (i) is trivial if $s = h$, because $\mathcal{G} \cong \mathcal{E}$ in this case. Thus we may assume that $h < s$. Set $a_0 = 0$ and

$$a_i := \ker(\mathcal{F}_i \cap \mathcal{G}) \text{ with } k_i := a_i - a_{i-1},$$

for $i = 1, \dots, s$. If we denote by $\mathcal{R}_i \subseteq \mathcal{F}_i/\mathcal{F}_{i-1}$ the image of $\mathcal{F}_i \cap \mathcal{G}$ by the quotient map $\pi_i : \mathcal{F}_i \rightarrow \mathcal{F}_i/\mathcal{F}_{i-1}$, then \mathcal{R}_i is trivial, i.e. $\mathcal{F}_i \cap \mathcal{G} \subseteq \mathcal{F}_{i-1}$, if and only if $k_i = 0$. Setting $S := \{i \in \{1, \dots, s\} \mid k_i > 0\}$, we have $\sum_{i=1}^s k_i = \sum_{i \in S} k_i = \rho$ and that $\mathcal{G} \cong \mathcal{F}_h$ if and only if $k_i = r_i$ for all $i \leq h$, or equivalently $k_i = 0$ for all $i > h$. Since \mathcal{F}_0 is trivial, we have $\mathcal{R}_1 \cong \mathcal{F}_1 \cap \mathcal{G}$. Thus we have $\deg \mathcal{G} = \sum_{i \in S} \deg \mathcal{R}_i$. Since each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable, we have $\deg \mathcal{R}_i \leq k_i \mu_i$ for all $i \in S$ and so we may use that $\mu_i > \mu_j$ for all $i < j$ to get part (i).

For part (ii) let $\mathcal{G} \subset \mathcal{E}$ be a rank k subsheaf of maximal degree and define k_i , $S \subseteq \{1, \dots, s\}$ and the sheaves $\mathcal{R}_i \subset \mathcal{F}_i/\mathcal{F}_{i-1}$ as above. Then we have $\sum_{i \in S} k_i = k$ and $\deg \mathcal{G} \leq \sum_{i \in S} k_i \mu_i$ and again we may use that $\mu_i > \mu_j$ for all $i < j$, to get the assertion. Part (iii) comes directly from the definition of $\delta_k(\mathcal{E})$. \square

As immediate corollaries of Theorem 4.11 we get the following.

Corollary 4.12. *We have $s_k(\mathcal{E}, \Phi) = s_k(\mathcal{E}) = s_k(\text{gr}(\mathcal{E}))$ for all k .*

Corollary 4.13. *For k with $r - r_s < k < r$, we have*

$$\delta_k(\mathcal{E}, \Phi) = \delta_k(\mathcal{E}) = d_1 + \dots + d_{s-1} + \delta_{k+r_s-r}(\mathcal{E}/\mathcal{F}_{s-1}).$$

4.2. Simplicity. Again let X be a smooth curve of genus g . Fix $\mathcal{R} \in \text{Pic}(X)$ and set $\gamma := \deg \mathcal{R}$. For a map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{R}$, set

$$\text{End}(\mathcal{E}, \Phi) := \{f \in \text{End}(\mathcal{E}) \mid \hat{f} \circ \Phi = \Phi \circ f\},$$

where \hat{f} is the map $f \otimes \text{id}_{\mathcal{R}} : \mathcal{E} \otimes \mathcal{R} \rightarrow \mathcal{E} \otimes \mathcal{R}$.

In case $\gamma > 0$, it often happens that $\text{End}(\mathcal{E}, \Phi)$ is properly contained in $\text{End}(\mathcal{E})$ and (\mathcal{E}, Φ) is simple with \mathcal{E} not simple, e.g. stable Higgs fields when $g \geq 2$ or stable co-Higgs fields when $g = 0$. In this short section, we consider the case $\gamma < 0$ and show why this is seldom the case for $\gamma < 0$.

We assume that $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with the Harder-Narasimhan filtration (2) and that $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{R}$ is nonzero and so $s \geq 2$. Note that every endomorphism of \mathcal{E} preserves the Harder-Narasimhan filtration of \mathcal{E} . By Remark 4.3, every endomorphism of (\mathcal{E}, Φ) also preserves the Harder-Narasimhan filtration of \mathcal{E} . Now set $\mathcal{K} := \ker(\Phi)$ and then we have $\mathcal{K} \supseteq \mathcal{F}_1$ by the case $i = 1$ of Remark 4.3 or Lemma 5.3 below. For two maps $\varphi \in \text{End}(\mathcal{E}/\mathcal{F}_{r-1})$ and $\psi \in \text{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{K})$, define a map $f : \mathcal{E} \rightarrow \mathcal{E}$ to be the following composition:

$$\mathcal{E} \rightarrow \mathcal{E}/\mathcal{F}_{r-1} \rightarrow \mathcal{K} \hookrightarrow \mathcal{E},$$

where the first map is the natural quotient and the second map is given by $\psi \circ \varphi$. By the definition of \mathcal{K} , we have $\Phi \circ f = 0$. If Φ is 2-nilpotent, i.e. $\text{Im}(\Phi) \subseteq \mathcal{K} \otimes \mathcal{R}$, e.g. if $s = 2$ by Lemma 4.1, we have $\hat{f} \circ \Phi = 0$. So, if Φ is 2-nilpotent and $\text{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{K}) \neq 0$, then we have $\text{End}(\mathcal{E}, \Phi) \neq \mathbb{C}$. We also see from $\mathcal{F}_1 \subseteq \mathcal{K}$ that if $\text{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{F}_1) \neq 0$, then we have $\text{End}(\mathcal{E}, \Phi) \neq \mathbb{C}$. By Riemann-Roch, we get $\text{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{F}_1) \neq 0$, if $d_s/r_s < d_1/r_1 + g - 1$. Since $\Phi \neq 0$ and each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable, we have $d_s/r_s \leq d_1/r_1 + \gamma$. Now if $\mathcal{R} \cong \mathcal{T}_D$, then we have $\gamma \leq 2 - 2g$ and so $d_s/r_s < g - 1 + d_1/r_1$ for all $g \geq 2$. Thus we get the following.

Proposition 4.14. *For $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with a nonzero co-Higgs field Φ on a smooth curve X of genus $g \geq 2$, the pair (\mathcal{E}, Φ) is not simple.*

Remark 4.15. In our set-up, adding a nonzero map Φ to an unstable bundle \mathcal{E} does not help enough to get a semistable pair (\mathcal{E}, Φ) ; usually it is not simple, e.g. any endomorphism inducing $\mathcal{F}_s \rightarrow \mathcal{F}_1$ commutes with Φ .

5. HIGHER DIMENSIONAL CASE

In this section we consider the case when the dimension of X is at least two. Note that a coherent sheaf \mathcal{E} on X is semistable if and only if $\mu_+(\mathcal{E}) = \mu_-(\mathcal{E})$.

5.1. Case of \mathcal{T}_D semistable. We fix a polarization $\mathcal{O}_X(1)$ with respect to which we consider slope, stability and semistability. We assume that \mathcal{T}_D is semistable with $\mu(\mathcal{T}_D) < 0$; in case $\mu(\mathcal{T}_D) \geq 0$, we would get that the framework would be the construction of stable or semistable co-Higgs or logarithmic co-Higgs bundles as in [4]. There are several manifolds X with \mathcal{T}_X semistable, or equivalently with the semistable cotangent bundle; see [26].

Choose a pair (\mathcal{E}, Φ) with \mathcal{E} a torsion-free sheaf of rank r and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_D$ with the Harder-Narasimhan filtration (2) of \mathcal{E} . Then the sheaf $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free semistable sheaf for all i and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i-1}/\mathcal{F}_{i-2})$ for every $i > 1$. As in §3 on curve case, for fixed integers r_i and d_i , we consider the set $\mathbb{U}_X(s; r_1, d_1, \dots, r_s, d_s)$ of torsion-free sheaves of rank r on X with the Harder-Narasimhan filtration (2) with subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ of ranks r_i and degrees d_i for $i = 1, \dots, s$.

Recall that in characteristic zero the tensor product of two semistable sheaves is still semistable by [22, Theorem 2.5]. So if Φ is not trivial, then we get $s \geq 2$ and so \mathcal{E} is not semistable with the Harder-Narasimhan filtration (4) for $\mathcal{E} \otimes \mathcal{T}_D$. If \mathcal{A} is a semistable torsion-free sheaf, then we have

$$\mu(\mathcal{A} \otimes \mathcal{T}_D) = \mu(\mathcal{A}) + \gamma.$$

Thus if $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ and there is a nonzero map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_D$, then we get $d_1/r_1 + \gamma \geq d_s/r_s$; see Corollary 3.9. Now let us use the same idea in Lemma

2.3. Define

$$\ell_2 = \ell_2(\mathcal{E}) := \max_{1 \leq i \leq s} \{ i \mid \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \gamma \geq \mu(\mathcal{F}_s/\mathcal{F}_{s-1}) \}$$

and then we have $\Phi(\mathcal{E}) \subset \mathcal{F}_{\ell_2} \otimes \mathcal{T}_{\mathcal{D}}$. From $\gamma < 0$, we get $\ell_2 \leq s-1$. On the other hand, letting

$$\ell_1 = \ell_1(\mathcal{E}) := \min_{1 \leq j \leq s} \{ j \mid \mu(\mathcal{F}_{\ell_2}/\mathcal{F}_{\ell_2-1}) + \gamma < \mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \},$$

the map Φ induces a nonzero map $\overline{\Phi} : \mathcal{E}/\mathcal{F}_{\ell_1} \rightarrow \mathcal{F}_{\ell_2} \otimes \mathcal{T}_{\mathcal{D}}$. In particular, if $\ell_1 \geq \ell_2$, e.g. $s = 2$ or $d_2/r_2 + \gamma < d_s/r_s$, which imply $\ell_2 = 1$, then any such map Φ is 2-nilpotent.

In [4, Section 2] we consider the following exact sequence for $r \geq 2$

$$(13) \quad 0 \rightarrow \mathcal{O}_X^{\oplus(r-1)} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{A} \rightarrow 0,$$

where \mathcal{A} is a line bundle of $\deg \mathcal{A} < 0$ with $h^0(\mathcal{T}_{\mathcal{D}} \otimes \mathcal{A}^\vee) \geq r-1$ and $Z \subset X$ is a locally complete intersection of codimension two. Under certain assumptions on Z , we may choose \mathcal{E} to be reflexive or locally free. Then any $(r-1)$ -dimensional linear subspace of $H^0(\mathcal{T}_{\mathcal{D}} \otimes \mathcal{A}^\vee)$ produces a nonzero 2-nilpotent co-Higgs field defined by the following composition:

$$\mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{A} \rightarrow \mathcal{T}_{\mathcal{D}}^{\oplus(r-1)} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}.$$

Assume now the existence of an endomorphism $v : \mathcal{E} \rightarrow \mathcal{E}$ such that $v' \circ \Phi = \Phi \circ v$, where $v' : \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ is the induces by v and the identity map on $\mathcal{T}_{\mathcal{D}}$. Since we assume that $\deg \mathcal{A} < 0$, (13) is the Harder-Narasimhan filtration of \mathcal{E} . We also assume that (13) does not split and so every automorphism of \mathcal{E} is induced by an element of $H^0(\mathcal{A}^\vee)^{\oplus(r-1)}$ and an $(r-1) \times (r-1)$ -matrix of constants acting on $\mathcal{O}_X^{\oplus(r-1)}$. Note that, if $r = 2$, these assumptions imply $h^0(\text{End}(\mathcal{E})) = 1 + h^0(\mathcal{A}^\vee)$. In this case, the co-Higgs field Φ is obtained by composing a map $\Phi_1 : \mathcal{I}_Z \otimes \mathcal{A} \rightarrow \mathcal{T}_{\mathcal{D}}$ with a map $\Phi_2 : \mathcal{T}_{\mathcal{D}} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ induced by the inclusion in (13).

5.2. Case of $\mathcal{T}_{\mathcal{D}}$ not semistable. In this subsection we assume that $\mathcal{T}_{\mathcal{D}}$ is not semistable so that it admits the Harder-Narasimhan filtration

$$(14) \quad \{0\} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_h = \mathcal{T}_{\mathcal{D}}$$

with $h \geq 2$. Assume further that $\mu_+(\mathcal{T}_{\mathcal{D}}) = \mu(\mathcal{H}_1) < 0$. Since $h \geq 2$, we have $\dim X \geq h \geq 2$.

Fix a torsion-free sheaf \mathcal{E} of rank r and degree d with Harder-Narasimhan filtration (2). We assume the existence of a nonzero logarithmic co-Higgs field $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$.

Lemma 5.1. *If \mathcal{E} is reflexive, then \mathcal{F}_i is also reflexive for each i .*

Proof. In case $n = 1$, the sheaf \mathcal{F}_i in (2) is locally free and in particular reflexive. Now assume $n \geq 2$ and then we need to prove that \mathcal{F}_i has depth at least two. This is true, because \mathcal{E} has depth at least two and $\mathcal{E}/\mathcal{F}_i$ has no torsion and so it has positive depth. \square

Remark 5.2. Lemma 5.1 works for arbitrary $\mathcal{T}_{\mathcal{D}}$, even in the case $n = 1$.

Lemma 5.3. *We have $\mathcal{F}_1 \subseteq \ker(\Phi)$ and $s \geq 2$.*

Proof. Assume $\Phi(\mathcal{F}_1) \neq 0$ and let i_0 be the minimal integer $i \in \{1, \dots, s\}$ such that $\Phi(\mathcal{F}_1) \subseteq \mathcal{F}_i \otimes \mathcal{T}_D$. By the definition of i_0 , the map Φ induces a nonzero map $\varphi : \mathcal{F}_1 \rightarrow (\mathcal{F}_{i_0}/\mathcal{F}_{i_0-1}) \otimes \mathcal{T}_D$. Since the tensor product of two semistable sheaves, modulo its torsion, is again semistable by [22, Theorem 2.5] and $\mu(\mathcal{H}_1) < 0$, the sheaf $gr((\mathcal{F}_{i_0}/\mathcal{F}_{i_0-1}) \otimes \mathcal{T}_D)$ given by the Harder-Narasimhan filtration of \mathcal{T}_D has all its factors with slope less than $\mu(\mathcal{F}_1)$. Thus we get $\Phi = 0$, a contradiction.

Now Φ is a nonzero map with $\ker(\Phi) \supseteq \mathcal{F}_1$ and so we have $s \geq 2$. \square

Remark 5.4. By Lemma 5.1, the pair $(\mathcal{F}_1, 0)$ is a logarithmic co-Higgs subsheaf of (\mathcal{E}, Φ) and so (\mathcal{E}, Φ) is not semistable. In particular, \mathcal{E} is also not semistable.

5.2.1. *Rank 2 case.* In this subsection we consider the co-Higgs sheaves (\mathcal{E}, Φ) with \mathcal{E} reflexive of rank two and Φ nonzero.

Lemma 5.5. *If \mathcal{E} is reflexive of rank two, then Φ is 2-nilpotent.*

Proof. Since Φ is nonzero, the sheaf \mathcal{F}_1 has rank one by Lemma 5.3. Since \mathcal{F}_1 is reflexive on a smooth variety X by Lemma 5.1, it is a line bundle by [16, Proposition 1.9]. Now we get that $\mathcal{E}/\mathcal{F}_1 \cong \mathcal{I}_Z \otimes \mathcal{A}$ for some line bundle \mathcal{A} and some closed subscheme $Z \subset X$ with $\dim Z \leq n-2$. By definition of Harder-Narasimhan filtration, we have $\deg \mathcal{A} < \deg \mathcal{F}_1$. Let $\psi : \mathcal{E} \rightarrow (\mathcal{E}/\mathcal{F}_1) \otimes \mathcal{T}_D$ be the map induced by Φ . Since $\ker(\Phi) \supseteq \mathcal{F}_1$ by Lemma 5.3, it is sufficient to prove that $\Phi(\mathcal{E}) \subseteq \mathcal{F}_1 \otimes \mathcal{T}_D$, i.e. $\psi = 0$. Note that ψ induces a map $\tilde{\psi} : (\mathcal{E}/\mathcal{F}_1) \rightarrow (\mathcal{E}/\mathcal{F}_1) \otimes \mathcal{T}_D$ with $\text{Im}(\psi) = \text{Im}(\tilde{\psi})$, due to $\mathcal{F}_1 \subseteq \ker(\Phi)$. Since $(\mathcal{E}/\mathcal{F}_1)$ has rank one and it is torsion-free, it is semistable. Again as in the proof of Lemma 5.3, since $\mu(\mathcal{H}_1) < 0$ and the tensor product of two semistable sheaves, modulo its torsion, is semistable by [22, Theorem 2.5], we have $\mu((\mathcal{E}/\mathcal{F}_1) \otimes \mathcal{H}_1) < \mu(\mathcal{E}/\mathcal{F}_1)$ and so $\tilde{\psi} = 0$. Thus we have $\psi = 0$. \square

Now we describe all pairs (\mathcal{E}, Φ) with \mathcal{E} reflexive of rank two and Φ nonzero. By Lemma 5.3 and assumption that Φ is nonzero, the sheaf \mathcal{E} is not semistable and $s = 2$. By Lemmas 5.1, 5.3, 5.5 and [16, Proposition 1.9], the map Φ is 2-nilpotent and it fits into an exact sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \det(\mathcal{E}) \otimes \mathcal{F}_1^\vee \rightarrow 0,$$

with Z a closed subscheme of X with either $Z = \emptyset$ or $\dim Z = n-2$. Moreover, Φ is uniquely determined by a map $u : \det(\mathcal{E}) \otimes \mathcal{F}_1^\vee \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_D$. Thus the set of all logarithmic co-Higgs structures on \mathcal{E} is parametrized by

$$V(\mathcal{E}) := H^0(\det(\mathcal{E})^\vee \otimes \mathcal{F}_1^{\otimes 2} \otimes \mathcal{T}_D).$$

The trivial element $0 \in V(\mathcal{E})$ corresponds to the trivial co-Higgs field $\Phi = 0$. Note that $\Phi = 0$ also exists for stable sheaves.

Now we reverse the construction. Fix two line bundles \mathcal{L}_1 and \mathcal{L}_2 on X with $\deg \mathcal{L}_1 > \deg \mathcal{L}_2$ and a closed subscheme $Z \subset X$ such that a general extension

$$(15) \quad 0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \otimes \mathcal{L}_2 \rightarrow 0$$

is reflexive. We just observed that any co-Higgs field $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_D$ is 2-nilpotent and that $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D) \cong H^0(\mathcal{L}_1 \otimes \mathcal{L}_2^\vee \otimes \mathcal{T}_D)$. We may see [16, Theorem 4.1] for a description about the conditions on \mathcal{L}_1 , \mathcal{L}_2 , ω_X and Z assuring the existence of a reflexive sheaf fitting in (15) when $n = 3$. Since (15) is the Harder-Narasimhan filtration of any \mathcal{E} fitting into (15), so the family of the co-Higgs sheaves (\mathcal{E}, Φ) with $gr(\mathcal{E}) = \mathcal{L}_1 \oplus (\mathcal{I}_Z \otimes \mathcal{L}_2)$ is parametrized by a fibration over $\mathbb{P}\text{Ext}_X^1(\mathcal{I}_Z \otimes \mathcal{L}_2, \mathcal{L}_1)$ whose fibre over $[\mathcal{E}]$ is $H^0(\mathcal{L}_1 \otimes \mathcal{L}_2^\vee \otimes \mathcal{T}_D)$.

Remark 5.6. Assume $s = 2$ and $\mu_+(\mathcal{T}_D) < 0$. For a torsion-free coherent sheaf \mathcal{E} of rank at least 2, as in the proof of Lemma 5.5 we see that every logarithmic co-Higgs field $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_D$ is integrable and 2-nilpotent with

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_D) \cong \mathrm{Hom}(\mathcal{E}/\mathcal{F}_1, \mathcal{F}_1 \otimes \mathcal{T}_D),$$

where \mathcal{F}_1 is semistable, and $\mathcal{E}/\mathcal{F}_1$ is torsion-free and semistable. Recall that if \mathcal{E} is reflexive, then so is \mathcal{F}_1 by Lemma 5.1. Take any exact sequence

$$(16) \quad 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{G} \rightarrow \mathcal{E}/\mathcal{F}_1 \rightarrow 0.$$

Any such extension in (16) is torsion-free. For any fixed \mathcal{G} fitting into (16), not necessarily reflexive, the proof of Lemma 5.5 shows that every logarithmic co-Higgs field $\Phi : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{T}_D$ is integrable and 2-nilpotent with $\mathrm{Hom}(\mathcal{G}, \mathcal{G} \otimes \mathcal{T}_D) \cong \mathrm{Hom}(\mathcal{E}/\mathcal{F}_1, \mathcal{F}_1 \otimes \mathcal{T}_D)$.

5.2.2. Rank 3 case. We assume $r = 3$ and that \mathcal{E} is reflexive. Since we assume $\mu_+(\mathcal{T}_D) < 0$, we get $s \geq 2$ by Lemma 5.3 and so $s \in \{2, 3\}$.

Remark 5.7. The case $s = 2$ is dealt in Remark 5.6. In this case, the sheaf \mathcal{F}_1 is either a line bundle or a semistable reflexive sheaf of rank two with $\mathcal{E}/\mathcal{F}_1 \cong \mathcal{I}_Z \otimes \mathcal{A}$ for some line bundle \mathcal{A} and a closed subscheme $Z \subset X$ with $\dim Z \leq n - 2$. In both cases, we may apply Remark 2.8.

From now on we assume $s = 3$ and so the sheaf \mathcal{F}_i in (2) has rank i for each i . By Lemma 5.1, the sheaf \mathcal{F}_1 is a line bundle and \mathcal{F}_2 is reflexive so that $\mathcal{F}_2/\mathcal{F}_1 \cong \mathcal{I}_{Z_1} \otimes \mathcal{A}_1$ and $\mathcal{E}/\mathcal{F}_2 \cong \mathcal{I}_{Z_2} \otimes \mathcal{A}_2$ with $\mathcal{A}_1, \mathcal{A}_2$ line bundles and Z_1, Z_2 closed subschemes of X with dimension at most $n - 2$. Here we have $\deg \mathcal{F}_1 > \deg \mathcal{A}_1 > \deg \mathcal{A}_2$. Set

$$\delta(\mathcal{E}) := \mu_-(\mathcal{F}_2) - \mu_+(\mathcal{E} \otimes \mathcal{T}_D),$$

where $\mu_-(\mathcal{F}_2) = \deg \mathcal{A}_1$ and $\mu_+(\mathcal{E} \otimes \mathcal{T}_D) = \deg \mathcal{F}_1 + \mu_+(\mathcal{T}_D)$

(a) Assume $\delta(\mathcal{E}) > 0$ and then we have $\Phi|_{\mathcal{F}_2} = 0$, i.e. Φ is uniquely induced by a map $u_1 : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \rightarrow \mathcal{E} \otimes \mathcal{T}_D$. Since $\mathcal{E}/\mathcal{F}_2 \cong \mathcal{I}_{Z_2} \otimes \mathcal{A}_2$ is of rank one and $\mu_+(\mathcal{T}_D) < 0$, the composition of u_1 with the quotient map $\mathcal{E} \otimes \mathcal{T}_D \rightarrow (\mathcal{E}/\mathcal{F}_2) \otimes \mathcal{T}_D$ is trivial, implying $\mathrm{Im}(u_1) \subseteq \mathcal{F}_2 \otimes \mathcal{T}_D$. Thus Φ is uniquely determined by a map $u : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \rightarrow \mathcal{F}_2 \otimes \mathcal{T}_D$. Conversely, any map $u : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \rightarrow \mathcal{F}_2 \otimes \mathcal{T}_D$ induces a 2-nilpotent logarithmic co-Higgs field on \mathcal{E} by taking the composition $u \circ \pi$, where $\pi : \mathcal{E} \rightarrow \mathcal{E}/\mathcal{F}_2$ is the quotient map.

(b) Assume now $\delta(\mathcal{E}) \leq 0$. Set $\mathcal{B} := \mathrm{Im}(\Phi|_{\mathcal{F}_2})$ and $\mathcal{G} := \mathrm{Im}(\Phi)$. Since we have

$$\mu_+(\mathcal{E}/\mathcal{F}_2 \otimes \mathcal{T}_D) = \mu(\mathcal{E}/\mathcal{F}_2) + \mu_+(\mathcal{T}_D) < \mu(\mathcal{E}/\mathcal{F}_2),$$

the composition of Φ with the quotient map $\mathcal{E} \otimes \mathcal{T}_D \rightarrow (\mathcal{E}/\mathcal{F}_2) \otimes \mathcal{T}_D$ is trivial and so we have $\mathcal{G} \subseteq \mathcal{F}_2 \otimes \mathcal{T}_D$. If \mathcal{B} is trivial, then we may apply part (a), i.e. Φ is 2-nilpotent and it is uniquely induced by $u : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \rightarrow \mathcal{F}_2 \otimes \mathcal{T}_D$. Now we assume that \mathcal{B} is not trivial. Since $\Phi(\mathcal{F}_1) = 0$ and $\mathcal{F}_2/\mathcal{F}_1$ is a torsion-free sheaf of rank one, we have $\mathcal{B} \cong \mathcal{F}_2/\mathcal{F}_1$ and so $\mathrm{rk}(\mathcal{G}) \in \{1, 2\}$. Note that we have $\mathcal{B} \subseteq \mathcal{F}_1 \otimes \mathcal{T}_D$ from $\mu_+(\mathcal{E}/\mathcal{F}_2) + \mu_+(\mathcal{T}_D) < \mu_+(\mathcal{E}/\mathcal{F}_2)$.

(b-i) First assume $\mathrm{rk}(\mathcal{G}) = 1$ and then \mathcal{B} is a subsheaf of \mathcal{G} with the same rank. Since $\mathcal{F}_1 \otimes \mathcal{T}_D$ is a saturated subsheaf of $\mathcal{F}_2 \otimes \mathcal{T}_D$, we have $\mathcal{G} \subseteq \mathcal{F}_1 \otimes \mathcal{T}_D$. Thus Φ is uniquely determined by a map $\mathcal{E}/\mathcal{F}_1 \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_D$, i.e. by an element of $H^0(\mathcal{T}_D \otimes \mathcal{F}_1 \otimes \mathcal{A}^\vee)$; the converse also holds, but we cannot guarantee the integrability of the associated logarithmic co-Higgs field.

(b-ii) Now assume $\text{rk}(\mathcal{G}) = 2$. Since we have $\mathcal{G} = \psi(\mathcal{E}/\mathcal{F}_1)$ for the map $\psi : \mathcal{E}/\mathcal{F}_1 \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, the map ψ is injective as a map of sheaves and $\mathcal{G} \cong \mathcal{E}/\mathcal{F}_1$. In this case we also have $\mathcal{F}_1 = \ker(\Phi)$. We get that \mathcal{E} is a reflexive sheaf fitting into an exact sequence

$$(17) \quad 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{E} \xrightarrow{f} \mathcal{G} \rightarrow 0$$

with \mathcal{F}_1 a line bundle and \mathcal{G} a torsion-free unstable sheaf of rank two with $\deg \mathcal{F}_1 > \mu_+(\mathcal{G})$. The map Φ is determined by a unique injective map $v : \mathcal{G} \rightarrow \mathcal{F}_2 \otimes \mathcal{T}_{\mathcal{D}}$. Conversely, set $\mathcal{G}_1 \subset \mathcal{G}$ to be the Harder-Narasimhan filtration of \mathcal{G} and $\mathcal{F}_2 = f^{-1}(\mathcal{G}_1)$, where f is the surjection in (17). Then the composition of the quotient map $\mathcal{E} \rightarrow \mathcal{E}/\mathcal{F}_1$ with an injective map $\mathcal{G} \rightarrow \mathcal{F}_2 \otimes \mathcal{T}_{\mathcal{D}}$ induces a logarithmic co-Higgs field Φ with the given data $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{G})$, which does not necessarily satisfy the integrability condition. Note that if $\mathcal{G} \subset \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$, i.e. Φ comes from an injective map $\mathcal{G} \rightarrow \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$, then Φ is 2-nilpotent and so it is integrable.

Example 5.8. Assume that $\mathcal{T}_{\mathcal{D}}$ is not semistable with Harder-Narasimhan filtration (14) and set $\mu_2(\mathcal{T}_{\mathcal{D}}) := \mu(\mathcal{H}_2/\mathcal{H}_1)$. Let \mathcal{E} be a torsion-free sheaf of rank r with (2) as its Harder-Narasimhan filtration and assume $\mu_+(\mathcal{E}) - \mu_-(\mathcal{E}) < \mu_2(\mathcal{T}_{\mathcal{D}})$. In this case, for any map $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, the sheaf $\text{Im}(\Phi)$ is contained in the subsheaf $\overline{\mathcal{E} \otimes \mathcal{H}_1}$ of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, which is the image of the natural map $\mathcal{E} \otimes \mathcal{H}_1 \rightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. We have $\overline{\mathcal{E} \otimes \mathcal{H}_1} \cong \mathcal{E} \otimes \mathcal{H}_1$ if either \mathcal{E} or \mathcal{H}_1 is locally free. Note that \mathcal{H}_1 is locally free, if it has rank one, because \mathcal{H}_1 is reflexive and X is smooth; see [16, Proposition 1.9]. In particular, if $n = 2$, then \mathcal{H}_1 is a line bundle and $\mu_2(\mathcal{T}_{\mathcal{D}}) = \mu_-(\mathcal{T}_{\mathcal{D}})$. Thus under these assumptions we may repeat the observations given in the case $\mathcal{T}_{\mathcal{D}}$ semistable using \mathcal{H}_1 instead of $\mathcal{T}_{\mathcal{D}}$. Without any assumption on $\mu_2(\mathcal{T}_{\mathcal{D}})$ we may see at least a part of the logarithmic co-Higgs fields of \mathcal{E} in this way.

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